

Optimal execution and block trade pricing: a general framework*

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Abstract

In this article, we develop a general CARA framework to study optimal execution and to price block trades. We prove existence and regularity results for optimal liquidation strategies and we provide several differential characterizations. We also give two different proofs that the usual restriction to deterministic liquidation strategies is optimal. In addition, we focus on the important topic of block trade pricing and we therefore give a price to financial (il)liquidity. In particular, we provide a closed-form formula for the price a block trade when there is no time constraint to liquidate, and a differential characterization in the time-constrained case. Numerical methods are eventually discussed.

Introduction

A general issue for stock traders consists in buying or selling large quantities of shares within a certain time. Unlike for small trades, a trader executing large blocks of shares cannot ignore the significant impact his orders have on the market. A trader willing to sell instantaneously a large quantity of shares would indeed incur a very high execution cost or could even be prevented to succeed in selling because of limited available liquidity.

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In a nutshell, traders face a trade-off between price risk on the one hand and both execution cost and market impact on the other hand. Traders liquidating too fast incur high execution costs but being too slow exposes to possible adverse price fluctuations, effectively leading to liquidation at lower-than-expected prices. For that reason, traders usually split their large orders into smaller ones to be executed progressively within a certain time window. Research on optimal execution¹ has initially been dedicated to this issue of optimally splitting those large orders.²

The above trade-off between execution cost, market impact and price risk first appeared in the economic and financial literature with Grinold and Kahn [15] and has been widely studied since then thanks to the framework developed by Almgren and Chriss in their two seminal papers [2, 3]. It is noteworthy that the literature on optimal liquidation started beforehand with Bertsimas and Lo [7], but their research focused on the minimization of expected execution costs only, consequently ignoring price risk that plays an important part in practice.

In the last ten years, the framework proposed by Almgren and Chriss has largely been used in practice and generalized either to better fit real market conditions or to enlarge the scope of modeling possibilities. Initially developed in discrete time with linear execution costs and within a Bachelier model for the price, it has also been considered in continuous time and generalized to allow for nonlinear execution costs and random execution costs [5].³ Black-Scholes dynamics for the price has also been considered⁴ and attempts to generalize the model in other directions have also been made, for instance to take account of stochastic volatility and liquidity [1]. Discussions on the optimization criterions and their consequences on optimal strategies are also very present in the literature (see for instance [4], [11], [20] and [26]). In this paper, we consider a Von Neumann-Morgenstern expected utility framework and the specific case of an investor with constant absolute risk aversion.⁵ This CARA framework has been studied in [24] in which the author prove⁶ that going

¹Optimal execution may sometimes be referred to as best execution or optimal liquidation. In this paper, we only focus on liquidation but the issues of buying large quantities of shares can be tackled in the same way.

²Today, new strands of academic research have developed. Following the seminal paper by Obizhaeva and Wang [21], many authors model the dynamics of the order book instead of having a statistical view on execution costs. Also, the focus of research has slightly moved from time scheduling to the actual way to proceed with execution. Liquidation with limit orders – see [6, 16, 17] – and dark pools – see [18, 19] – are now important topics.

³The proofs in [5] are however limited to the case of a C^2 execution cost function and C^2 trajectories... two unrealistic assumptions as we shall see in the text.

⁴For short periods of time there is no real difference between Bachelier dynamics and Black-Scholes dynamics.

⁵Very interesting results in the case of IARA and DARA utility functions are presented in [23]. For practical applications, choosing the appropriate level of risk aversion is already a complex task and we shall not go beyond CARA utility functions.

⁶We repeat below their proof that deterministic strategies are optimal. Another proof is also

from deterministic liquidation strategies to stochastic liquidation strategies does not bring any advantage.

In this paper, we develop a framework that is general enough to cover most nonlinear execution cost functions used in practice. In addition to nonlinearity, we cover the case of (deterministic) time-dependent execution costs ; an important case to account for the shape of market volume curves.⁷ However, as far as price dynamics is concerned, we focused on the case of a Bachelier model with a drift given by a linear permanent market impact.⁸ Optimal liquidation strategies are discussed, along with the important problem of block trade pricing. The academic literature indeed focuses on the rhythm at which liquidation should be carried out but ignores the global cost of liquidation. We regard this issue as an important one, both to know at what price to buy a large portfolio and for risk managers to evaluate liquidity premia.

We present the general framework in Section 1. Section 2 presents existence, uniqueness and regularity results for the optimal deterministic liquidation strategy. Differential characterizations are provided that are useful when it comes to numerics. In addition, we provide a general proof of the reduction of the problem to a first order ODE when market volume is constant. Section 3 gives a first proof that no stochastic liquidation strategy can improve the best deterministic strategy. Section 4 is dedicated to block trade pricing. We first prove regularity results for the value function.⁹ We then determine the price of a block of q_0 shares given market conditions and the fact that the shares have to be liquidated within a certain time window. The partial differential equation associated to the pricing problem (and to the initial liquidation problem) has a singularity at final time and we provide a way to approximate the price of a block trade using a relaxed problem that does not have any singularity. Interestingly, we also provide a closed-form expression for the price of a block trade when no time window is imposed for liquidation afterwards. Section 5 uses the relaxed problem of Section 4 in which liquidation is not enforced to provide another proof that no stochastic strategy can do better than the best deterministic strategy. Finally, Section 6 is dedicated to a discussion about numerical methods to find the optimal liquidation strategy and/or the price of a block trade. Numerical methods are rarely discussed in the literature (see [12] for one of the rare examples) but we shall see that some natural discretizations may not provide satisfying results.

provided in Section 5 using a completely different approach.

⁷The available liquidity is not the same at each hour of the day. In Europe, in addition to the importance of the near-opening and near-closing periods, the opening of the New York marketplace is an important event.

⁸The choice of a Bachelier model over a Black-Scholes model is mainly dictated by mathematical considerations. However, there is no real difference from the modeling side because liquidation problems are short-term problems.

⁹In [24], the authors discuss the properties of the value function. However, contrary to what they claim, the only thing they prove is almost everywhere differentiability and that the Hamilton-Jacobi is solved almost everywhere.

1 Setup and notations

A trader with a portfolio containing¹⁰ $q_0 > 0$ shares of a given stock is willing to liquidate his portfolio over a time window $[0, T]$.

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions. We assume that all stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$.

We introduce the set $\mathcal{P}(t, T)$ of progressively measurable processes defined on $[t, T]$ and the set $\mathcal{A} = \left\{ (v_t)_{t \in [0, T]} \in \mathcal{P}(0, T), (t, \omega) \mapsto \int_0^t v_s(\omega) ds \in L^\infty([0, T] \times \Omega) \right\}$. For a control process $(v_t)_{t \in [0, T]} \in \mathcal{A}$ – representing the velocity at which the trader sells his shares –, we denote by $(q_t)_{t \in [0, T]}$ the number of shares in the portfolio where q_t is given by:

$$q_t = q_0 - \int_0^t v_s ds.$$

The trades impact the market price in two distinct ways. Firstly, there is a permanent market impact (assumed to be linear¹¹) that imposes a drift to the price process $(S_t)_{t \in [0, T]}$:

$$dS_t = \sigma dW_t - kv_t dt, \quad \sigma > 0, k \geq 0.$$

Secondly, the price obtained by the trader at time t is not S_t because of execution costs. To model these execution costs we introduce a function $L \in C^1(\mathbb{R}, \mathbb{R})$ verifying the following hypotheses:¹²

- $L(0) = 0$,
- L is an even function,
- L is increasing on \mathbb{R}_+ ,
- L is strictly convex,
- $\exists \epsilon > 0, \exists A, a > 0, B, b \geq 0, \forall \rho \in \mathbb{R}, L(\rho) \geq A|\rho|^{1+\epsilon} - B$ and $|L'(\rho)| \leq a|\rho|^\epsilon + b$.

¹⁰The case $q_0 < 0$ can be treated using the same tools.

¹¹See [13].

¹²We want to cover the cases $L(\rho) = \eta|\rho|^{1+\phi}$ for $\eta > 0$ and $\phi > 0$. In particular, the function L is not C^2 when $\phi < 1$ (a common case when the function L is calibrated on real data), and although it is a C^2 function when $\phi > 1$, it does not verify in this latter case that $L'' > 0$.

We also introduce a market volume process $(V_t)_{t \in [0, T]}$ assumed to be continuous, deterministic and such that $\exists \underline{V} > 0, \overline{V} > 0, \forall t \in [0, T], \underline{V} \leq V_t \leq \overline{V}$.

This allows, for $v \in \mathcal{A}$, to define the cash process¹³ $(X_t)_{t \in [0, T]}$ as:

$$X_t = \int_0^t \left(v_s S_s - V_s L \left(\frac{v_s}{V_s} \right) - \psi |v_s| \right) ds,$$

where the execution cost is divided into two parts: a linear part which represents a fixed cost ($\psi \geq 0$) per share – linked to the bid-ask spread and the tick size –, and a strictly convex and superlinear part modeled by L , to account for the cost of trading fast.

We define our objective function for $v \in \mathcal{A}$ as

$$J(v) = \mathbb{E} [-\exp(-\gamma X_T)],$$

where $\gamma > 0$ is the absolute risk aversion parameter of the trader.

The goal of this paper is to solve two problems:

- We want to find an optimal strategy to liquidate the portfolio within a certain time T . In other words, we want to maximize J over $\mathcal{A}_0 = \left\{ v \in \mathcal{A}, \int_0^T v_s ds = q_0 \right\}$.
- We want to give a price to a block trade. In other words we want to determine the maximal price an agent should buy a portfolio of q_0 shares of the stock, given that this portfolio will have to be liquidated on the market.

2 Optimal trading curves

2.1 Introduction

In this section, we are going to consider the case of deterministic strategies for optimal liquidation. Deterministic liquidation strategies are simpler to analyze than general stochastic ones because the cash process at the terminal time T (that is X_T) is normally distributed when the liquidation strategy $(v_t)_{t \in [0, T]}$ is deterministic. As we shall see in Section 3, it turns out that no stochastic strategy can improve the best deterministic strategy and that the restriction to deterministic strategies is therefore not a real restriction.

To start, let us define the set \mathcal{A}_{det} of deterministic strategies in \mathcal{A} and the set $\mathcal{A}_{0, \text{det}}$ of deterministic (liquidation) strategies in \mathcal{A}_0 .

¹³The cash process may be equal to $-\infty$.

Depending on whether v is in \mathcal{A} , \mathcal{A}_{\det} or $\mathcal{A}_{0,\det}$, we have different expression and properties for the cash process. The following proposition and its corollary state these properties.

Proposition 1. *Let us consider $v \in \mathcal{A}$ and $t \in [0, T]$. We have:*

$$X_t + q_t S_t - \frac{k}{2} q_t^2 = q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^t |v_s| ds + \int_0^t \sigma q_s dW_s.$$

In particular, if $v \in \mathcal{A}_{\det}$, if $\int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds < +\infty$, and if $\int_0^t |v_s| ds < +\infty$ then,

$$X_t + q_t S_t - \frac{k}{2} q_t^2 \sim \mathcal{N}\left(q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^t |v_s| ds, \sigma^2 \int_0^t q_s^2 ds\right).$$

Proof:

By definition:

$$\begin{aligned} X_t &= \int_0^t v_s S_s ds - \int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^t |v_s| ds \\ &= q_0 S_0 - q_t S_t - k \int_0^t v_s q_s ds + \int_0^t \sigma q_s dW_s - \int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^t |v_s| ds \\ &= -q_t S_t + \frac{k}{2} q_t^2 + q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^t |v_s| ds + \int_0^t \sigma q_s dW_s. \end{aligned}$$

When $(q_s)_{s \in [0, t]}$ is deterministic, if $\int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds < +\infty$ and $\int_0^t |v_s| ds < +\infty$, then $X_t + q_t S_t - \frac{k}{2} q_t^2$ is normally distributed with mean $q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^t |v_s| ds$ and variance $\sigma^2 \int_0^t q_s^2 ds$. \square

A straightforward corollary of Proposition 1 is that X_T is normally distributed in the deterministic case when $q_T = 0$. More precisely:

Corollary 1 (Distribution of X_T). *Let us consider $v \in \mathcal{A}_{0,\det}$. If $\int_0^t V_s L\left(\frac{v_s}{V_s}\right) ds < +\infty$, and $\int_0^t |v_s| ds < +\infty$, then:*

$$X_T \sim \mathcal{N}\left(q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds, \sigma^2 \int_0^T q_s^2 ds\right).$$

Using the above corollary and the Laplace transform of a normal distribution, we can find a closed-form expression for the objective function J :

Corollary 2. *Let us consider $v \in \mathcal{A}_{0,\det}$. Then:*

$$J(v) = -\exp \left(-\gamma \left(q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^T V_s L \left(\frac{v_s}{V_s} \right) ds - \psi \int_0^t |v_s| ds - \frac{1}{2} \gamma \sigma^2 \int_0^T q_s^2 ds \right) \right).$$

2.2 Existence, uniqueness and characterization of a minimizer

We can then define a new objective function for $v \in \mathcal{A}_{0,\det}$ by

$$I(v) = \int_0^T V_s L \left(\frac{v_s}{V_s} \right) ds + \psi \int_0^t |v_s| ds + \frac{1}{2} \gamma \sigma^2 \int_0^T q_s^2 ds,$$

so that:

$$J(v) = -\exp \left(-\gamma \left(q_0 S_0 - \frac{k}{2} q_0^2 - I(v) \right) \right).$$

Then

$$\sup_{v \in \mathcal{A}_{0,\det}} J(v) = -\exp \left(-\gamma \left(q_0 S_0 - \frac{k}{2} q_0^2 - \inf_{v \in \mathcal{A}_{0,\det}} I(v) \right) \right),$$

and the maximizers of J correspond to the minimizers of I .

We are going to study the function I but instead of regarding I as a function of v , we regard I as a function of q . More precisely, we define:

$$\begin{aligned} \mathcal{I}_\psi &: W_{q_0,0}^{1,1+\epsilon}(0,T) \rightarrow \mathbb{R}_+ \\ q &\mapsto \int_0^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \psi |\dot{q}(s)| + \frac{1}{2} \gamma \sigma^2 q^2(s) \right) ds, \end{aligned}$$

where the Sobolev space $W_{a_1,a_2}^{1,1+\epsilon}(t_1,t_2)$ is the set of functions $q \in W^{1,1+\epsilon}(t_1,t_2)$ satisfying $q(t_1) = a_1$ and $q(t_2) = a_2$. It is a natural space for our problem because of the assumptions on L .

The following theorem states that there exists a unique minimizer of \mathcal{I}_ψ in $W_{q_0,0}^{1,1+\epsilon}(0,T)$.

Theorem 1 (Existence and uniqueness of a minimizer). *There exists a unique minimizer $q^* \in W_{q_0,0}^{1,1+\epsilon}(0,T)$ of the function \mathcal{I}_ψ . This minimizer is a nonnegative and decreasing function.*

Proof:

Let us first consider the function $q_{line} \in W_{q_0,0}^{1,1+\epsilon}(0,T)$ defined by $q_{line}(t) = q_0 \left(1 - \frac{t}{T} \right)$. $\mathcal{I}_\psi(q_{line}) < +\infty$ and therefore $\inf_{W_{q_0,0}^{1,1+\epsilon}(0,T)} \mathcal{I}_\psi < +\infty$.

Now, let us consider a sequence $(q_n)_{n \in \mathbb{N}}$ of functions in $W_{q_0,0}^{1,1+\epsilon}(0,T)$ such that $\lim_{n \rightarrow +\infty} \mathcal{I}_\psi(q_n) = \inf_{W_{q_0,0}^{1,1+\epsilon}(0,T)} \mathcal{I}_\psi$.

For $n \in \mathbb{N}$, we can define \tilde{q}_n by $\tilde{q}_n(t) = \inf_{s \leq t} (q_n(s))_+$. We have $\tilde{q}_n \in W_{q_0,0}^{1,1+\epsilon}(0,T)$ with $\dot{\tilde{q}}_n = \dot{q}_n 1_{\{q_n > 0, q_n = \tilde{q}_n\}}$. Hence, using the monotonicity property of L , we obtain that $\mathcal{I}_\psi(\tilde{q}_n) \leq \mathcal{I}_\psi(q_n)$ and we therefore have that $\lim_{n \rightarrow +\infty} \mathcal{I}_\psi(\tilde{q}_n) = \inf_{W_{q_0,0}^{1,1+\epsilon}(0,T)} \mathcal{I}_\psi$.

Now, we have:

$$\int_0^T \left(A \frac{|\dot{\tilde{q}}_n(s)|^{1+\epsilon}}{V_s^\epsilon} - BV_s \right) ds \leq \int_0^T V_s L \left(\frac{\dot{\tilde{q}}_n(s)}{V_s} \right) ds \leq \mathcal{I}_\psi(\tilde{q}_n).$$

Hence $\sup_{n \in \mathbb{N}} \|\dot{\tilde{q}}_n\|_{L^{1+\epsilon}(0,T)} < +\infty$. Now, since $|\tilde{q}_n| \leq q_0$, we know that $(\tilde{q}_n)_{n \in \mathbb{N}}$ is a bounded sequence of $W_{q_0,0}^{1,1+\epsilon}(0,T)$. Hence, there exists $\tilde{q} \in W_{q_0,0}^{1,1+\epsilon}(0,T)$ such that, up to a subsequence, $\tilde{q}_n \rightarrow \tilde{q}$ in $C^0([0,T])$ and $\dot{\tilde{q}}_n \rightharpoonup \dot{\tilde{q}}$ in $L^{1+\epsilon}(0,T)$.

Now, $\left| L' \left(\frac{\dot{\tilde{q}}(s)}{V_s} \right) \right| \leq a \left| \frac{\dot{\tilde{q}}(s)}{V_s} \right|^\epsilon + b$ so that the function $s \mapsto L' \left(\frac{\dot{\tilde{q}}(s)}{V_s} \right)$ is in $L^{1+\frac{1}{\epsilon}}(0,T)$.

Hence, $\int_0^T L' \left(\frac{\dot{\tilde{q}}(s)}{V_s} \right) (\dot{\tilde{q}}_n(s) - \dot{\tilde{q}}(s)) ds \rightarrow 0$ and we have using the convexity of L that:

$$\begin{aligned} \mathcal{I}_\psi(\tilde{q}) &= \int_0^T \left(V_s L \left(\frac{\dot{\tilde{q}}(s)}{V_s} \right) + \psi |\dot{\tilde{q}}(s)| + \frac{1}{2} \gamma \sigma^2 \tilde{q}^2(s) \right) ds \\ &= \int_0^T V_s L \left(\frac{\dot{\tilde{q}}(s)}{V_s} \right) ds + \psi q_0 + \frac{1}{2} \gamma \sigma^2 \int_0^T \tilde{q}^2(s) ds \\ &\leq \int_0^T V_s L \left(\frac{\dot{\tilde{q}}_n(s)}{V_s} \right) ds - \int_0^T L' \left(\frac{\dot{\tilde{q}}(s)}{V_s} \right) (\dot{\tilde{q}}_n(s) - \dot{\tilde{q}}(s)) ds \\ &\quad + \psi q_0 + \frac{1}{2} \gamma \sigma^2 \int_0^T \tilde{q}^2(s) ds. \end{aligned}$$

Hence:

$$\begin{aligned} \mathcal{I}_\psi(\tilde{q}) &\leq \liminf_{n \rightarrow +\infty} \int_0^T V_s L \left(\frac{\dot{\tilde{q}}_n(s)}{V_s} \right) ds + \psi q_0 + \frac{1}{2} \gamma \sigma^2 \int_0^T \tilde{q}^2(s) ds \\ &\leq \liminf_{n \rightarrow +\infty} \int_0^T V_s L \left(\frac{\dot{\tilde{q}}_n(s)}{V_s} \right) ds + \psi q_0 + \frac{1}{2} \gamma \sigma^2 \lim_{n \rightarrow +\infty} \int_0^T \tilde{q}_n^2(s) ds \\ &\leq \liminf_{n \rightarrow +\infty} \mathcal{I}_\psi(\tilde{q}_n) \\ &\leq \inf_{W_{q_0,0}^{1,1+\epsilon}(0,T)} \mathcal{I}_\psi. \end{aligned}$$

Hence $q^* = \tilde{q}$ is a minimizer of \mathcal{I}_ψ and, as announced, it is a nonnegative and decreasing function (as the limit of a sequence of such functions).

Coming to uniqueness, if there were two different minimizers, q^* and q^{**} , then we would have $\mathcal{I}_\psi \left(\frac{q^* + q^{**}}{2} \right) < \frac{1}{2} \mathcal{I}_\psi(q^*) + \frac{1}{2} \mathcal{I}_\psi(q^{**}) = \mathcal{I}_\psi(q^*)$ because L is convex and

$q \mapsto q^2$ is strictly convex, in contradiction with the optimality of q^* . Hence, q^* is unique. \square

Now, let us notice that if $q \in W_{q_0,0}^{1,1+\epsilon}(0,T)$ is decreasing, then:

$$\begin{aligned}\mathcal{I}_\psi(q) &= \int_0^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \psi |\dot{q}(s)| + \frac{1}{2} \gamma \sigma^2 q^2(s) \right) ds \\ &= \int_0^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q^2(s) \right) ds + \psi q_0 \\ &= \mathcal{I}_0(q) + \psi q_0.\end{aligned}$$

Hence, q^* is the unique minimizer of \mathcal{I}_0 and it does not depend on ψ . From now, we shall refer to \mathcal{I}_0 as \mathcal{I} and q^* will be the unique minimizer of \mathcal{I} .

Using this property of q^* , we are going to derive two different differential characterizations of q^* and eventually prove that $q^* \in C^1([0,T])$.

We first exhibit the Euler-Lagrange equation for q^* :

Proposition 2 (Euler-Lagrange equation for q^*). *q^* is the unique weak solution of the differential equation:*

$$\frac{d}{dt} L' \left(\frac{\dot{q}(t)}{V_t} \right) = \gamma \sigma^2 q(t), \quad q(0) = q_0, \quad q(T) = 0.$$

Proof:

For $\phi \in C^1([0,T])$ with $\phi(0) = \phi(T) = 0$, we introduce $i_{q,\phi} : \epsilon \in \mathbb{R} \mapsto \mathcal{I}(q + \epsilon\phi)$.

The optimality of q^* implies that $i'_{q^*,\phi}(0) = 0$, that is:

$$\int_0^T \left(L' \left(\frac{\dot{q}(t)}{V_t} \right) \phi'(t) + \gamma \sigma^2 q(t) \phi(t) \right) dt = 0.$$

This is exactly the weak formulation of the Euler-Lagrange equation.

Now, if q is a weak solution of the Euler-Lagrange equation, because i is convex, we have:

$$\mathcal{I}(q^*) = i_{q,q^*-q}(1) \geq i_{q,q^*-q}(0) + i'_{q,q^*-q}(0) = \mathcal{I}(q).$$

Hence, $q = q^*$. \square

Associated to this Euler-Lagrange characterization, we exhibit the Hamiltonian characterization of q^* . For that purpose, we introduce H , the Legendre transform of L : $H(p) = \sup_{\rho \in \mathbb{R}} \rho p - L(\rho)$, and we recall that the strict convexity of L implies that H is a C^1 function.

Proposition 3 (Hamiltonian equations for q^*). *Let us introduce $p^*(t) = L' \left(\frac{\dot{q}^*(t)}{V_t} \right)$. Then, (p^*, q^*) solves the following system of equations:*

$$\begin{cases} \dot{p}(t) &= \gamma \sigma^2 q(t) \\ \dot{q}(t) &= V_t H'(p(t)) \end{cases} \quad q(0) = q_0, \quad q(T) = 0.$$

Moreover, if there exists a couple (p, q) of two C^1 functions satisfying the above equations, then $q = q^*$.

Proof:

By duality, $p^*(t) = L' \left(\frac{\dot{q}^*(t)}{V_t} \right) \implies \frac{\dot{q}^*(t)}{V_t} = H'(p^*(t)) \implies \dot{q}^*(t) = V_t H'(p^*(t))$.

Now, if (p, q) is a couple of C^1 functions satisfying the above equations, then, by duality, $p(t) = L' \left(\frac{\dot{q}(t)}{V_t} \right)$. Hence:

$$\frac{d}{dt} L' \left(\frac{\dot{q}(t)}{V_t} \right) = \gamma \sigma^2 q(t), \quad q(0) = q_0, \quad q(T) = 0,$$

and the conclusion follows by the above Proposition. \square

A consequence of this differential characterization is the following:

Corollary 3. $q^* \in C^1([0, T])$.

We shall see in the next subsection that there exists cases where $q^* \notin C^2([0, T])$. This means that one has to be careful when considering the Euler-Lagrange equation.

Now, since q^* is C^1 , $v^* = -\dot{q}^* \in \mathcal{A}_{0,\det}$, we can gather all the results to obtain a theorem about the minimizers of J :

Theorem 2 (Existence and uniqueness of a minimizer for J in $\mathcal{A}_{0,\det}$). *There exists a unique minimizer v^* of J in $\mathcal{A}_{0,\det}$.*

Moreover, $t \mapsto v_t^$ is a continuous and nonnegative function.*

2.3 The special case of a flat market volume curve

In this subsection, we consider the special case $V_t = V$. In that case, and under smoothness assumptions on L , the Euler-Lagrange equation can be simplified and boils down to a first order ODE.

Theorem 3 (First order ODE for the optimal liquidation strategy). *Let us assume that $V_t = V$. Moreover, let us suppose¹⁴ that $L \in C^2(\mathbb{R}^*)$ with $\forall \rho \in \mathbb{R}^*, L''(\rho) > 0$. Let us introduce the strictly increasing function $r : \rho \in \mathbb{R}_+ \mapsto \rho L'(\rho) - L(\rho)$.*

- *If $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is not integrable in $\xi = 0$ then there exists $C > 0$ such that q^* is uniquely characterized by:*

$$\dot{q}^*(t) = -Vr^{-1} \left[\frac{\gamma\sigma^2}{2V} q^*(t)^2 + C \right] \quad \text{with} \quad q^*(T) = 0.$$

- *If $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is integrable in $\xi = 0$, then let us define q_0^{\lim} by:*

$$\begin{cases} q_0^{\lim} = +\infty, & \text{if } \int_0^\infty \frac{1}{Vr^{-1}\left(\frac{\gamma\sigma^2}{2V}\xi^2\right)} d\xi < T \\ \int_0^{q_0^{\lim}} \frac{1}{Vr^{-1}\left(\frac{\gamma\sigma^2}{2V}\xi^2\right)} d\xi = T, & \text{otherwise.} \end{cases}$$

- *For $q_0 > q_0^{\lim}$, there exists $C > 0$ such that q^* is uniquely characterized by:*

$$\dot{q}^*(t) = -Vr^{-1} \left[\frac{\gamma\sigma^2}{2V} q^*(t)^2 + C \right] \quad \text{with} \quad q^*(T) = 0.$$

- *For $q_0 \leq q_0^{\lim}$, q^* is uniquely characterized by:*

$$\dot{q}^*(t) = -Vr^{-1} \left[\frac{\gamma\sigma^2}{2V} q^*(t)^2 \right] \quad \text{with} \quad q^*(0) = q_0.$$

Moreover, if $q \in C^1([0, T])$ with $q(0) = q_0$ and $q(T) = 0$ solves, for some constant $C \geq 0$, the equation:

$$\dot{q}(t) = -Vr^{-1} \left[\frac{\gamma\sigma^2}{2V} q(t)^2 + C \right],$$

then $q = q^*$.

Proof:

The function $r : \rho \in \mathbb{R}_+ \mapsto \rho L'(\rho) - L(\rho)$ verifies:

- $r(0) = 0$,
- r is strictly increasing (because L is strictly convex),
- r is continuous,

¹⁴This hypothesis is verified by the execution cost functions we mentioned in the introduction. In particular, it is important to exclude $\rho = 0$ in the assumption because $L(\rho) = \eta|\rho|^{1+\phi}$ is not C^2 in 0.

- r is C^1 on $(0, +\infty)$, with $r'(\rho) = \rho L''(\rho)$,
- $\lim_{\rho \rightarrow +\infty} r(\rho) = +\infty$,¹⁵
- r is a bijection from \mathbb{R}_+ to \mathbb{R}_+ ,
- r^{-1} is continuous,
- r^{-1} is C^1 on $(0, +\infty)$.

Now, let us consider a function $q \in C^1([0, T])$ with $q(0) = q_0$, $q(T) = 0$ and satisfying for some constant $C \geq 0$:

$$\dot{q}(t) = -Vr^{-1} \left[\frac{\gamma\sigma^2}{2V} q(t)^2 + C \right].$$

We argue that $q = q^*$. To prove this let us consider two cases:

Case (i): $\forall t < T, q(t) > 0$.

In that case, q is C^2 on $[0, T)$, $\dot{q} < 0$ on $[0, T)$, and we can differentiate the equation $r \left(-\frac{\dot{q}(t)}{V} \right) = \frac{\gamma\sigma^2}{2V} q(t)^2 + C$, to obtain for $t \in [0, T)$:

$$\begin{aligned} -\frac{\ddot{q}(t)}{V} r' \left(-\frac{\dot{q}(t)}{V} \right) &= \frac{\gamma\sigma^2}{V} q(t) \dot{q}(t) \\ \frac{\ddot{q}(t) \dot{q}(t)}{V^2} L'' \left(-\frac{\dot{q}(t)}{V} \right) &= \frac{\gamma\sigma^2}{V} q(t) \dot{q}(t). \end{aligned}$$

Dividing by $\frac{\dot{q}(t)}{V}$, we get:

$$\frac{\ddot{q}(t)}{V} L'' \left(-\frac{\dot{q}(t)}{V} \right) = \gamma\sigma^2 q(t),$$

i.e.:

$$-\frac{d}{dt} L' \left(-\frac{\dot{q}(t)}{V} \right) = \gamma\sigma^2 q(t).$$

Because L is even, this is equivalent to the Euler-Lagrange equation:

$$\frac{d}{dt} L' \left(\frac{\dot{q}(t)}{V} \right) = \gamma\sigma^2 q(t).$$

We then conclude that $q = q^*$.

Case (ii): $\exists \tau < T, q(\tau) = 0$ and $\forall t < \tau, q(t) > 0$.

¹⁵If indeed r was bounded then $\forall \rho \geq 1, \left(\frac{L(\rho)}{\rho} \right)' = \frac{r(\rho)}{\rho^2} \leq \frac{\|r\|_\infty}{\rho^2}$. Hence, $\forall \rho \geq 1, \frac{L(\rho)}{\rho} \leq L(1) + \|r\|_\infty$, contrary to our assumptions on L .

In that case the first thing to notice is that necessarily $C = 0$ and $q(t) = 0, \forall t \in [\tau, T]$. We know indeed that q is decreasing (with $\dot{q} \leq -Vr^{-1}(C)$) and that $q(T) = 0$ so that q must remain equal to zero once it touches 0.

Now, we can apply the reasoning we applied for case (i) to the interval $[0, \tau]$. Hence, $\forall t \in [0, \tau]$:

$$\frac{d}{dt}L' \left(\frac{\dot{q}(t)}{V} \right) = \gamma\sigma^2 q(t).$$

Now, because $q = 0$ on $(\tau, T]$, the above equation also holds on $(\tau, T]$. Consequently, q is a weak solution to the Euler-Lagrange equation and therefore $q = q^*$.

Now, we need to prove that $\forall q_0 > 0, \exists C \geq 0, \exists q \in C^1([0, T])$ with $q(0) = q_0$ and $q(T) = 0$ such that:

$$\dot{q}(t) = -Vr^{-1} \left[\frac{\gamma\sigma^2}{2V} q(t)^2 + C \right].$$

For that purpose, let us introduce the following backward Cauchy problem in $[0, T]$ for a given constant C :

$$(\mathcal{E}_C) \quad \dot{q}(t) = -Vr^{-1} \left[\frac{\gamma\sigma^2}{2V} q(t)^2 + C \right] \quad \text{with} \quad q(T) = 0$$

$t \mapsto q(t) = 0$ is a solution of the equation (\mathcal{E}_0) . However, this solution is not unique whenever $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is integrable in $\xi = 0$.

If $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is integrable in $\xi = 0$, let us implicitly define the function q^{lim} by:

$$\int_0^{q^{\text{lim}}(t)} \frac{dq}{Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^2 \right)} = T - t,$$

for $t \in \left(T - \int_0^\infty \frac{dq}{Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^2 \right)}, T \right]$.

This function defines a solution to (\mathcal{E}_0) .

If $\int_0^\infty \frac{dq}{Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^2 \right)} \leq T$, then for all positive q_0 , we can consider the equation

$$(\mathcal{E}'_{q_0}) \quad \dot{q}(t) = -Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q(t)^2 \right) \quad \text{with} \quad q(0) = q_0$$

We know that $\xi \geq 0 \mapsto -Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} \xi^2 \right)$ is continuous and decreasing. Therefore, using Cauchy-Peano theorem for existence and monotonicity for uniqueness, there exists a unique solution q to (\mathcal{E}'_{q_0}) on some interval $[0, t_{q_0}[$. Because this solution is

a decreasing function, it either merges with q^{lim} at some point¹⁶ $t < T$ and is equal to 0 at time T , or it reaches 0 for some $t \in [0, T]$ (and stays equal to 0 after time t). Hence, the solution q of (\mathcal{E}'_{q_0}) is a solution to (\mathcal{E}_0) on $[0, T]$ and solves our problem.

Now, if $\int_0^\infty \frac{dq}{Vr^{-1}\left(\frac{\gamma\sigma^2}{2V}q^2\right)} > T$ then q^{lim} is defined on $[0, T]$ and the above reasoning applies but only for $q_0 \leq q^{\text{lim}}(0)$.

The problem for a given $q_0 > q^{\text{lim}}(0)$ is then to find $C > 0$ so that a solution of (\mathcal{E}_C) reaches q_0 in $t = 0$. To this purpose, we define, for $C > 0$, the following family of equations:

$$(\mathcal{E}''_{C,q_0}) \quad \dot{q}(t) = -Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q(t)^2 + C \right) \quad \text{with} \quad q(0) = q_0$$

Let us consider for a given $C > 0$ the maximal solution of (\mathcal{E}''_{C,q_0}) (the Cauchy problem is well-posed because $\xi \mapsto -Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} \xi^2 + C \right)$ is locally Lipschitz). Since $\dot{q}(t) \leq -Vr^{-1}(C) < 0$, we know that there exists t_C so that $q(t_C) = 0$. Now, since q is decreasing, we can write

$$t_C = \int_0^{q_0} \frac{dq}{Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^2 + C \right)}.$$

$C \mapsto t_C$ is continuous with:

$$\lim_{C \rightarrow 0} t_C = \int_0^{q(0)} \frac{dq}{Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^2 \right)} > \int_0^{q^{\text{lim}}(0)} \frac{dq}{Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^2 \right)} = T,$$

and

$$\lim_{C \rightarrow +\infty} t_C = 0.$$

Hence, there exists $C > 0$ so that $t_C = T$. For this C , the solution q of (\mathcal{E}''_{C,q_0}) solves (\mathcal{E}_C) with $q(0) = q_0$.

Our problem is then solved in the case where $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is integrable in $\xi = 0$.

Now, coming to the case where $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is not integrable in $\xi = 0$, we can proceed as above. If we indeed fix $q_0 > 0$ and consider the family of equations (\mathcal{E}''_{C,q_0}) , we know that for any $C > 0$ there exists t_C so that the solution q of (\mathcal{E}''_{C,q_0}) verifies $q(t_C) = 0$.

¹⁶In fact it cannot merge at time $t < T$ because the backward Cauchy problem associated to the junction point would be well-posed since $\xi \mapsto -Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} \xi^2 \right)$ is locally Lipschitz on \mathbb{R}_+^* .

Since q is decreasing we can write

$$t_C = \int_0^{q_0} \frac{dq}{Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^2 + C \right)}.$$

$C \mapsto t_C$ is continuous. Since $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is not integrable in $\xi = 0$ we have $\lim_{C \rightarrow 0} t_C = +\infty$. Also, as above, $\lim_{C \rightarrow +\infty} t_C = 0$.

Hence, there exists $C > 0$ so that $t_C = T$. For this C , the solution q of (\mathcal{E}_{C,q_0}'') solves (\mathcal{E}_C) with $q(0) = q_0$. \square

This simple first order ODE can be used to find in closed-form the optimal liquidation strategy in at least two cases. The first case corresponds to the cost function $L(\rho) = \eta\rho^2$ introduced by Almgren and Chriss [3]. In that case we have the following proposition:

Proposition 4 (The Almgren-Chriss case). *Let us consider $L(\rho) = \eta\rho^2$. If $V_t = V$ then q^* is given by:*

$$q^*(t) = q_0 \frac{\sinh \left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}} (T - t) \right)}{\sinh \left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}} T \right)}.$$

Proof:

This is a pure application of the preceding theorem with $C = \frac{\gamma\sigma^2}{2V} \frac{q_0^2}{\sinh^2 \left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}} T \right)}$. \square

The second case corresponds to $L(\rho) = \eta\rho^{2+\delta}$, $(\delta > 0)$, for q_0 small.

Proposition 5 (Super-quadratic execution costs for small inventories). *Let us consider $L(\rho) = \eta\rho^{2+\delta}$ for $\delta > 0$. Let us assume that $V_t = V$.*

Then, for $q_0 \leq \left(\frac{\delta}{2+\delta} \right)^{\frac{2+\delta}{\delta}} T^{\frac{2+\delta}{\delta}} V^{\frac{1+\delta}{\delta}} \left(\frac{\gamma\sigma^2}{2\eta(1+\delta)} \right)^{\frac{1}{\delta}}$, q^ is given by:*

$$q^*(t) = \left(q_0^{\frac{\delta}{2+\delta}} - \frac{\delta}{2+\delta} V^{\frac{1+\delta}{2+\delta}} \left(\frac{\gamma\sigma^2}{2\eta(1+\delta)} \right)^{\frac{1}{2+\delta}} t \right)_+^{\frac{2+\delta}{\delta}}.$$

Proof:

This is a straightforward application of the preceding theorem with $C = 0$, the constant q_0^{\lim} being equal to $\left(\frac{\delta}{2+\delta} \right)^{\frac{2+\delta}{\delta}} T^{\frac{2+\delta}{\delta}} V^{\frac{1+\delta}{\delta}} \left(\frac{\gamma\sigma^2}{2\eta(1+\delta)} \right)^{\frac{1}{\delta}}$. \square

The latter case exemplifies the limited smoothness of q^* since q^* is C^1 but not C^2 .

3 Deterministic strategies versus stochastic strategies

In the above section, we only considered deterministic strategies. We now provide a proof that no stochastic strategy can improve the best deterministic one. This simple proof is based on the use of Girsanov Theorem and has been proposed in [24]. We repeat the argument here for the sake of completeness.

Theorem 4 (Optimality of deterministic strategies).

$$\sup_{v \in \mathcal{A}_0} \mathbb{E} [-\exp(-\gamma X_T)] = \sup_{v \in \mathcal{A}_{0,\det}} \mathbb{E} [-\exp(-\gamma X_T)].$$

Proof:

For any $v \in \mathcal{A}_0$, we know that

$$X_T = q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds - \psi \int_0^T |v_s| ds + \int_0^T \sigma q_s dW_s.$$

Hence:

$$\begin{aligned} \mathbb{E} [-\exp(-\gamma X_T)] &= -\exp\left(-\gamma\left(q_0 S_0 - \frac{k}{2} q_0^2\right)\right) \\ &\times \mathbb{E} \left[\exp\left(\gamma\left(\int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds + \psi \int_0^T |v_s| ds\right)\right) \exp\left(-\gamma \sigma \int_0^T q_s dW_s\right) \right]. \end{aligned}$$

Hence, if we introduce the probability measure \mathbb{Q} defined by the Radon-Nikodym derivative¹⁷

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\gamma \sigma \int_0^T q_s dW_s - \frac{1}{2} \gamma^2 \sigma^2 \int_0^T q_s^2 ds\right),$$

then:

$$\begin{aligned} \mathbb{E} [-\exp(-\gamma X_T)] &= -\exp\left(-\gamma\left(q_0 S_0 - \frac{k}{2} q_0^2\right)\right) \\ &\times \mathbb{E}^{\mathbb{Q}} \left[\exp\left(\gamma\left(\int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds + \psi \int_0^T |v_s| ds\right)\right) \exp\left(\frac{1}{2} \gamma^2 \sigma^2 \int_0^T q_s^2 ds\right) \right]. \end{aligned}$$

Hence:

$$\mathbb{E} [-\exp(-\gamma X_T)] = \mathbb{E}^{\mathbb{Q}} \left[-\exp\left(-\gamma\left(q_0 S_0 - \frac{k}{2} q_0^2 - \mathcal{I}_\psi(q)\right)\right) \right].$$

Now, let us fix $\omega \in \Omega$. If $t \mapsto q_t(\omega) \in W^{1,1+\epsilon}$, then $\mathcal{I}_\psi(q(\omega)) \geq \mathcal{I}_\psi(q^*)$. Otherwise, $\mathcal{I}_\psi(q(\omega)) = +\infty$ and we have $\mathcal{I}_\psi(q(\omega)) \geq \mathcal{I}_\psi(q^*)$.

¹⁷We can apply Girsanov theorem since q is bounded.

This leads to

$$\mathbb{E}[-\exp(-\gamma X_T)] \leq -\exp\left(-\gamma\left(q_0 S_0 - \frac{k}{2} q_0^2 - \mathcal{I}_\psi(q^*)\right)\right),$$

i.e.:

$$\mathbb{E}[-\exp(-\gamma X_T)] \leq \sup_{v \in \mathcal{A}_{0,\det}} \mathbb{E}[-\exp(-\gamma X_T)].$$

We then obtain

$$\sup_{v \in \mathcal{A}_0} \mathbb{E}[-\exp(-\gamma X_T)] \leq \sup_{v \in \mathcal{A}_{0,\det}} \mathbb{E}[-\exp(-\gamma X_T)].$$

Since the converse inequality holds, the result is proved. \square

This theorem states that no stochastic liquidation strategy can do better than v^* , not only in $\mathcal{A}_{0,\det}$ but more generally in \mathcal{A}_0 . Hence, we solved our first problem and we will present, in Section 6, numerical methods to approximate the optimal liquidation strategy.

We will also present another proof of the result of Theorem 4 based on a completely different idea in Section 5, more precisely using a relaxed problem to remove the singularity

4 Block trade pricing

In addition to the optimal liquidation strategy, we are interested in the determination of a price for a block trade. Given the state of the market, we want to determine the maximum price at which an agent would be ready to buy a portfolio containing q_0 shares when he is forced to liquidate the portfolio on the market within a certain time T . This maximum price $P(T, q_0, S_0)$, in our expected utility framework, is obtained by the indifference pricing approach:

$$P(T, q_0, S_0) = -\frac{1}{\gamma} \log \left(-\sup_{v \in \mathcal{A}_0} J(v) \right) = q_0 S_0 - \frac{k}{2} q_0^2 - \psi q_0 - \mathcal{I}(q^*),$$

where q^* is the optimal trajectory associated to the liquidation of a portfolio with q_0 shares on the interval $[0, T]$, as in Section 2.

In practice, one can first compute the optimal liquidation strategy and then compute the price $P(T, q_0, S_0)$. This is a two-step approach. Another approach consists in computing directly the price of a block trade. This one-step approach (or direct approach) is based on the value function of the optimal control problem of Section 2 and relies on the theory of first order Hamilton-Jacobi equations.

We introduce the value function of the control problem of Section 2:

$$\theta_T(\hat{t}, \hat{q}) = \inf_{q \in W_{\hat{q}, 0}^{1, 1+\epsilon}(\hat{t}, T)} \int_{\hat{t}}^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q^2(s) \right) ds.$$

Since we bounded ourself earlier to the case of positive inventories, this function is a priori defined on $[0, T) \times \mathbb{R}_+^*$. We rather consider it on $[0, T) \times \mathbb{R}$ – and we notice that $\forall \hat{t} \in [0, T)$, $\theta_T(\hat{t}, 0) = 0$ and $\forall (\hat{t}, \hat{q}) \in [0, T) \times \mathbb{R}$, $\theta_T(\hat{t}, -\hat{q}) = \theta_T(\hat{t}, \hat{q})$, because L is even.

The introduction of θ_T allows to write $P(T, q_0, S_0)$ as $q_0 S_0 - \frac{k}{2} q_0^2 - \psi q_0 - \theta_T(0, q_0)$. More generally, we can write the price $P(t, q, S)$ to liquidate q shares on a time window of length t when the price of the stock is S as

$$P(t, q, S) = qS - \frac{k}{2} q^2 - \psi q - \theta_T(T - t, q), \forall T \geq t$$

We then see that the study of block trade pricing boils down to the study of the value functions θ_T 's.

4.1 Study of θ_T

We are now going to study the properties of θ_T . For that purpose, let us denote $q_{\hat{t}, \hat{q}}^* \in W_{\hat{q}, 0}^{1, 1+\epsilon}(\hat{t}, T)$ the function at which the infimum in the definition of $\theta_T(\hat{t}, \hat{q})$ is attained. Such a function does exist by Theorem 1 since we can replace the couple $(0, q_0)$ by the couple (\hat{t}, \hat{q}) – and it is a C^1 function on $[\hat{t}, T]$.

A first important result is the behavior of $\theta_T(\hat{t}, \hat{q})$ when $\hat{t} \rightarrow T$. Proposition 6 states indeed that θ_T has a singularity at time T , because of the liquidation constraint $q_T = 0$.

Proposition 6 (Singularity of θ_T).

$$\lim_{\hat{t} \rightarrow T} \theta_T(\hat{t}, \hat{q}) = \begin{cases} 0, & \text{if } \hat{q} = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof:

$\theta_T(\hat{t}, 0) = 0$. The result is then obvious for $\hat{q} = 0$.

Now, using the properties of L , we get:

$$\begin{aligned}
\theta_T(\hat{t}, \hat{q}) &\geq \int_{\hat{t}}^T V_s L\left(\frac{\dot{q}_{\hat{t}, \hat{q}}^*(s)}{V_s}\right) ds \\
&\geq \int_{\hat{t}}^T \underline{V} L\left(\frac{\dot{q}_{\hat{t}, \hat{q}}^*(s)}{\underline{V}}\right) ds \\
&\geq (T - \hat{t}) \underline{V} L\left(\int_{\hat{t}}^T \frac{\dot{q}_{\hat{t}, \hat{q}}^*(s)}{\underline{V}} \frac{ds}{T - \hat{t}}\right) \\
&\geq \underline{V}(T - \hat{t}) L\left(\frac{\hat{q}}{\underline{V}(T - \hat{t})}\right)
\end{aligned}$$

Hence, if $\hat{q} \neq 0$, the superlinearity of L gives $\lim_{\hat{t} \rightarrow T} \theta_T(\hat{t}, \hat{q}) = +\infty$. \square

The singularity exhibited in Proposition 6, along with the limited regularity of L , make the use of classical theorems impossible as far as the regularity of θ_T is concerned. We shall prove using tools from convex analysis and results on semi-concave functions (with nonlinear modulus of semi-concavity) that θ_T is in fact C^1 on $[0, T) \times \mathbb{R}$ and solves a Hamilton-Jacobi equation in the classical sense.

As a first step, we prove that θ_T is locally Lipschitz on $[0, T) \times \mathbb{R}$ and therefore continuous on $[0, T) \times \mathbb{R}$.

In order to do that, let us prove a lemma that provides a bound to the gradient of $q_{\hat{t}, \hat{q}}^*$.

Lemma 1.

$$\forall Q > 0, \forall \epsilon \in (0, T), \exists C(Q, \epsilon) > 0, \forall (\hat{t}, \hat{q}) \in [0, T - \epsilon] \times [-Q, Q], \sup_{t \in [\hat{t}, T]} |\dot{q}_{\hat{t}, \hat{q}}^*(t)| \leq C(Q, \epsilon).$$

Proof:

We consider the case $\hat{q} \geq 0$. The case of a negative \hat{q} works exactly the same.

Using the same method as for Proposition 3, we know that $q_{\hat{t}, \hat{q}}^*$ solves an ODE of the form:

$$\begin{cases} \dot{p}(t) &= \gamma \sigma^2 q(t) \\ \dot{q}(t) &= V_t H'(p(t)) \end{cases} \quad q(\hat{t}) = \hat{q}, \quad q(T) = 0$$

where p is a C^1 function.

From the ODE, we know that $\forall t \in [\hat{t}, T], p(t) \in [p(\hat{t}), p(\hat{t}) + \gamma \sigma^2 (T - \hat{t}) \hat{q}]$. This gives $\forall t \in [\hat{t}, T], \dot{q}_{\hat{t}, \hat{q}}^*(t) \in [\underline{V} H'(p(\hat{t})), \min(0, \underline{V} H'(p(\hat{t}) + \gamma \sigma^2 (T - \hat{t}) \hat{q}))]$.

Now, if $H'(p(\hat{t}) + \gamma\sigma^2(T - \hat{t})\hat{q}) \geq 0$, then $H'(p(\hat{t}) + \gamma\sigma^2 QT) \geq 0$ and $p(\hat{t})$ is bounded from below by a constant that depends on Q .

Otherwise, we have:

$$-\hat{q} = \int_{\hat{t}}^T \dot{q}_{\hat{t}, \hat{q}}^*(t) dt \leq \underline{V} H'(p(\hat{t}) + \gamma\sigma^2(T - \hat{t})\hat{q})(T - \hat{t}) \leq \underline{V} H'(p(\hat{t}) + \gamma\sigma^2 QT)(T - \hat{t}).$$

Hence $H'(p(\hat{t}) + \gamma\sigma^2 QT) \geq -\frac{\hat{q}}{\underline{V}(T - \hat{t})} \geq -\frac{Q}{\epsilon \underline{V}}$ and $p(\hat{t})$ is bounded from below by a constant that depends on Q and ϵ .

Now, in general, because $\forall t \in [\hat{t}, T]$, $\dot{q}_{\hat{t}, \hat{q}}^*(t) \geq \overline{V} H'(p(\hat{t}))$, we have a lower bound for $\dot{q}_{\hat{t}, \hat{q}}^*(t)$. 0 being an upper bound, we have proved the result. \square

Given $Q > 0$ and $\epsilon \in (0, T)$, this lemma and the results of Section 2 allow to redefine $\theta_T(\hat{t}, \hat{q})$ for $(\hat{t}, \hat{q}) \in [0, T - \epsilon] \times [-Q, Q]$ by:

$$\theta_T(\hat{t}, \hat{q}) = \inf_{q \in C^1([\hat{t}, T]), q(\hat{t}) = \hat{q}, q(T) = 0, |\dot{q}(t)| \leq C(Q, \epsilon)} \int_{\hat{t}}^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q^2(s) \right) ds.$$

Using this new definition, we can prove that θ_T is locally Lipschitz on $[0, T] \times \mathbb{R}$:

Proposition 7 (Local Lipschitz property). $\forall Q > 0, \forall \epsilon \in (0, T)$, θ_T is Lipschitz on $[0, T - \epsilon] \times [-Q, Q]$.

Proof:

Let us consider (\hat{t}, \hat{q}) and (\tilde{t}, \tilde{q}) in $[0, T - \epsilon] \times [-Q, Q]$.

We have:

$$\theta_T(\hat{t}, \hat{q}) = \int_{\hat{t}}^T \left(V_s L \left(\frac{\dot{q}_{\hat{t}, \hat{q}}^*(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q_{\hat{t}, \hat{q}}^*(s)^2 \right) ds.$$

Denoting $q(t) = q_{\hat{t}, \hat{q}}^*(t) + \frac{T-t}{T-\hat{t}}(\tilde{q} - \hat{q})$, we have:

$$\theta_T(\hat{t}, \tilde{q}) \leq \int_{\hat{t}}^T \left(V_s L \left(\frac{\dot{q}_{\hat{t}, \hat{q}}^*(s) - \frac{\tilde{q} - \hat{q}}{T - \hat{t}}}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 \left(q_{\hat{t}, \hat{q}}^*(s) + \frac{T-s}{T-\hat{t}}(\tilde{q} - \hat{q}) \right)^2 \right) ds.$$

Hence:

$$\theta_T(\hat{t}, \tilde{q}) - \theta_T(\hat{t}, \hat{q}) \leq \int_{\hat{t}}^T V_s \left(L \left(\frac{\dot{q}_{\hat{t}, \hat{q}}^*(s) - \frac{\tilde{q} - \hat{q}}{T - \hat{t}}}{V_s} \right) - L \left(\frac{\dot{q}_{\hat{t}, \hat{q}}^*(s)}{V_s} \right) \right) ds$$

$$+\frac{1}{2}\gamma\sigma^2\int_{\hat{t}}^T\left(\left(q_{\hat{t},\hat{q}}^*(s)+\frac{T-s}{T-\hat{t}}(\tilde{q}-\hat{q})\right)^2-q_{\hat{t},\hat{q}}^*(s)^2\right)ds.$$

Because L is C^1 , it is locally Lipschitz and there exists therefore a constant $K(Q, \epsilon)$ such that:

$$\forall s \in [\hat{t}, T], L\left(\frac{\dot{q}_{\hat{t},\hat{q}}^*(s)-\frac{\tilde{q}-\hat{q}}{T-\hat{t}}}{V_s}\right)-L\left(\frac{\dot{q}_{\hat{t},\hat{q}}^*(s)}{V_s}\right)\leq K(Q, \epsilon)\frac{1}{V}\frac{|\tilde{q}-\hat{q}|}{T-\hat{t}}.$$

Similarly, there exists a constant $K(Q)$ such that:

$$\forall s \in [\hat{t}, T], \left(q_{\hat{t},\hat{q}}^*(s)+\frac{T-s}{T-\hat{t}}(\tilde{q}-\hat{q})\right)^2-q_{\hat{t},\hat{q}}^*(s)^2\leq K(Q)|\tilde{q}-\hat{q}|.$$

Hence, there exists a constant $K'(Q, \epsilon)$ such that

$$\theta_T(\hat{t}, \tilde{q})-\theta_T(\hat{t}, \hat{q})\leq K'(Q, \epsilon)|\tilde{q}-\hat{q}|T.$$

Reversing the roles of \hat{q} and \tilde{q} we eventually obtain that:

$$|\theta_T(\hat{t}, \hat{q})-\theta_T(\hat{t}, \tilde{q})|\leq K'(Q, \epsilon)|\tilde{q}-\hat{q}|T.$$

Now, if $\hat{t} < \tilde{t}$:

$$\theta_T(\hat{t}, \hat{q})=\int_{\hat{t}}^{\tilde{t}}\left(V_sL\left(\frac{\dot{q}_{\hat{t},\hat{q}}^*(s)}{V_s}\right)+\frac{1}{2}\gamma\sigma^2q_{\hat{t},\hat{q}}^*(s)^2\right)ds+\theta_T(\tilde{t}, q_{\hat{t},\hat{q}}^*(\tilde{t})).$$

Since $\dot{q}_{\hat{t},\hat{q}}^*$ and $q_{\hat{t},\hat{q}}^*$ are uniformly bounded, there exists a constant $M(Q, \epsilon)$ such that:

$$\begin{aligned}\theta_T(\hat{t}, \hat{q})&\leq M(Q, \epsilon)(\tilde{t}-\hat{t})+\theta_T(\tilde{t}, q_{\hat{t},\hat{q}}^*(\tilde{t}))\leq M(Q, \epsilon)(\tilde{t}-\hat{t})+\theta_T(\tilde{t}, \hat{q})+K'(Q, \epsilon)|q_{\hat{t},\hat{q}}^*(\tilde{t})-\hat{q}|T\\ &\leq M(Q, \epsilon)(\tilde{t}-\hat{t})+\theta_T(\tilde{t}, \hat{q})+K'(Q, \epsilon)C(Q, \epsilon)|\tilde{t}-\hat{t}|T.\end{aligned}$$

Hence, there exists a constant $K''(Q, \epsilon)$ such that:

$$0\leq\theta_T(\hat{t}, \hat{q})-\theta_T(\tilde{t}, \hat{q})\leq K''(Q, \epsilon)|\tilde{t}-\hat{t}|.$$

Combining the results we obtain the Lipschitz property. \square

Now, we exhibit the Hamilton-Jacobi equation solved by θ_T :

Proposition 8 (Hamilton-Jacobi equation). *θ_T is a viscosity solution of the Hamilton-Jacobi equation:*

$$-\partial_t\theta_T(t, q)-\frac{1}{2}\gamma\sigma^2q^2+V_tH(\partial_q\theta_T(t, q))=0, \quad \text{on } [0, T)\times\mathbb{R}.$$

Proof:

We start with the fact that θ_T is a subsolution of the Hamilton-Jacobi equation on $(0, T) \times \mathbb{R}$.

Let $(t, q) \in (0, T) \times \mathbb{R}$ and let $\varphi \in C^1((0, T) \times \mathbb{R})$ be such that $\theta_T - \varphi$ has a local maximum in (t, q) . We can then find $\tau > 0$ and $\eta > 0$ such that $\forall (t', q') \in [t - \tau, t + \tau] \times [q - \eta, q + \eta]$:

$$\theta_T(t, q) - \varphi(t, q) \geq \theta_T(t', q') - \varphi(t', q').$$

Now, let us consider $v \in \mathbb{R}$. If $h \in (0, \tau]$ is small enough, we have that $t + h < T$ and $q - vh \in [q - \eta, q + \eta]$. Then:

$$\begin{aligned} \varphi(t + h, q - vh) - \varphi(t, q) &\geq \theta_T(t + h, q - vh) - \theta_T(t, q) \\ \frac{\varphi(t + h, q - vh) - \varphi(t, q)}{h} &\geq -\frac{1}{h} \int_t^{t+h} \left(V_s L \left(\frac{v}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 (q - v(s - t))^2 \right) ds. \end{aligned}$$

Hence if $h \rightarrow 0$, we get:

$$\partial_t \varphi(t, q) - v \partial_q \varphi(t, q) \geq -V_t L \left(\frac{v}{V_t} \right) - \frac{1}{2} \gamma \sigma^2 q^2.$$

Hence, $\forall v \in \mathbb{R}$,

$$-\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + v \partial_q \varphi(t, q) - V_t L \left(\frac{v}{V_t} \right) \leq 0.$$

This gives:

$$\begin{aligned} -\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + \sup_{v \in \mathbb{R}} v \partial_q \varphi(t, q) - V_t L \left(\frac{v}{V_t} \right) &\leq 0 \\ -\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t \sup_{v \in \mathbb{R}} \frac{v}{V_t} \partial_q \varphi(t, q) - L \left(\frac{v}{V_t} \right) &\leq 0 \\ -\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H(\partial_q \varphi(t, q)) &\leq 0. \end{aligned}$$

This proves that θ_T is a subsolution of the Hamilton-Jacobi equation on $(0, T) \times \mathbb{R}$.

We now prove that θ_T is a supersolution of the Hamilton-Jacobi equation on $(0, T) \times \mathbb{R}$.

Let $(t, q) \in (0, T) \times \mathbb{R}$ and let $\varphi \in C^1((0, T) \times \mathbb{R})$ be such that $\theta_T - \varphi$ has a

local minimum in (t, q) . We can then find $\tau > 0$ and $\eta > 0$ such that $\forall (t', q') \in [t - \tau, t + \tau] \times [q - \eta, q + \eta]$:

$$\theta_T(t, q) - \varphi(t, q) \leq \theta_T(t', q') - \varphi(t', q').$$

$\forall \epsilon > 0, \forall h \in (0, \min(\tau, \frac{T-t}{2}))$, $\exists (v_s^{\epsilon, h})_{s \in [t, T]} \in \mathcal{A}_{0, \det}(t, q)$ such that:

$$\theta_T(t, q) - \theta_T\left(t + h, q_{t+h}^{v^{\epsilon, h}}\right) \geq \int_t^{t+h} \left(V_s L\left(\frac{v_s^{\epsilon, h}}{V_s}\right) + \frac{1}{2} \gamma \sigma^2 (q_s^{v^{\epsilon, h}})^2 \right) ds - \epsilon h,$$

where $q_s^{v^{\epsilon, h}} = q - \int_t^s v_w^{\epsilon, h} dw$.

Now, because of Lemma 1, we can always suppose that $|v^{\epsilon, h}|$ is uniformly bounded, independently of ϵ and h , by a constant M . Hence, if we have $h < \tau$, $h < \frac{T-t}{2}$, and $Mh < \eta$, then $q_{t+h}^{v^{\epsilon, h}} \in [q - \eta, q + \eta]$ and:

$$\varphi(t, q) - \varphi\left(t + h, q_{t+h}^{v^{\epsilon, h}}\right) \geq \int_t^{t+h} \left(V_s L\left(\frac{v_s^{\epsilon, h}}{V_s}\right) + \frac{1}{2} \gamma \sigma^2 (q_s^{v^{\epsilon, h}})^2 \right) ds - \epsilon h.$$

This gives:

$$\begin{aligned} & - \int_t^{t+h} \left(\partial_t \varphi\left(s, q_s^{v^{\epsilon, h}}\right) - v_s^{\epsilon, h} \partial_q \varphi\left(s, q_s^{v^{\epsilon, h}}\right) \right) ds \\ & \geq \int_t^{t+h} \left(V_s L\left(\frac{v_s^{\epsilon, h}}{V_s}\right) + \frac{1}{2} \gamma \sigma^2 (q_s^{v^{\epsilon, h}})^2 \right) ds - \epsilon h. \end{aligned}$$

Hence:

$$\begin{aligned} & \int_t^{t+h} \left(-\partial_t \varphi\left(s, q_s^{v^{\epsilon, h}}\right) - \frac{1}{2} \gamma \sigma^2 (q_s^{v^{\epsilon, h}})^2 + v_s^{\epsilon, h} \partial_q \varphi\left(s, q_s^{v^{\epsilon, h}}\right) - V_s L\left(\frac{v_s^{\epsilon, h}}{V_s}\right) \right) ds \geq -\epsilon h \\ & \int_t^{t+h} \left(-\partial_t \varphi\left(s, q_s^{v^{\epsilon, h}}\right) - \frac{1}{2} \gamma \sigma^2 (q_s^{v^{\epsilon, h}})^2 + V_s H\left(\partial_q \varphi\left(s, q_s^{v^{\epsilon, h}}\right)\right) \right) ds \geq -\epsilon h \\ & \frac{1}{h} \int_t^{t+h} \left(-\partial_t \varphi\left(s, q_s^{v^{\epsilon, h}}\right) - \frac{1}{2} \gamma \sigma^2 (q_s^{v^{\epsilon, h}})^2 + V_s H\left(\partial_q \varphi\left(s, q_s^{v^{\epsilon, h}}\right)\right) \right) ds \geq -\epsilon. \end{aligned}$$

Using, the continuity of $\partial_t \varphi$, $\partial_q \varphi$, and H , and the uniform bound on $v^{\epsilon, h}$, we then obtain, sending h to 0, that:

$$-\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H(\partial_q \varphi(t, q)) \geq -\epsilon.$$

This being true for any $\epsilon > 0$, we have that:

$$-\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H(\partial_q \varphi(t, q)) \geq 0$$

This proves that θ_T is a supersolution of the Hamilton-Jacobi equation on $(0, T) \times \mathbb{R}$.

Consequently, θ_T is a viscosity solution of the Hamilton-Jacobi equation on $(0, T) \times \mathbb{R}$. We need to prove now that the result holds on $[0, T) \times \mathbb{R}$.

Let $q \in \mathbb{R}$ and let $\varphi \in C^1([0, T) \times \mathbb{R})$ be such that $\theta_T - \varphi$ has a strict local maximum in $(0, q)$. Then, for $\epsilon > 0$ sufficiently small, $(t, q) \in (0, T) \times \mathbb{R} \mapsto \theta_T(t, q) - \varphi(t, q) - \frac{\epsilon}{t}$ has a local maximum in $(t_\epsilon, q_\epsilon) \in (0, T) \times \mathbb{R}$, and $\lim_{\epsilon \rightarrow 0} (t_\epsilon, q_\epsilon) = (0, q)$. Using what we already proved, we obtain:

$$-\partial_t \varphi(t_\epsilon, q_\epsilon) + \frac{\epsilon}{t_\epsilon^2} - \frac{1}{2} \gamma \sigma^2 q_\epsilon^2 + V_{t_\epsilon} H(\partial_q \varphi(t_\epsilon, q_\epsilon)) \leq 0.$$

Hence:

$$-\partial_t \varphi(t_\epsilon, q_\epsilon) - \frac{1}{2} \gamma \sigma^2 q_\epsilon^2 + V_{t_\epsilon} H(\partial_q \varphi(t_\epsilon, q_\epsilon)) \leq 0,$$

and sending ϵ to 0 we obtain:

$$-\partial_t \varphi(0, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_0 H(\partial_q \varphi(0, q)) \leq 0.$$

Therefore, θ_T is a subsolution of the Hamilton-Jacobi equation on $[0, T) \times \mathbb{R}$.

Similarly, let $q \in \mathbb{R}$ and let $\varphi \in C^1([0, T) \times \mathbb{R})$ be such that $\theta_T - \varphi$ has a strict local minimum in $(0, q)$. Then, for $\epsilon > 0$ sufficiently small, $(t, q) \in (0, T) \times \mathbb{R} \mapsto \theta_T(t, q) - \varphi(t, q) + \frac{\epsilon}{t}$ has a local minimum in $(t_\epsilon, q_\epsilon) \in (0, T) \times \mathbb{R}$, and $\lim_{\epsilon \rightarrow 0} (t_\epsilon, q_\epsilon) = (0, q)$. Using what we already proved we obtain:

$$-\partial_t \varphi(t_\epsilon, q_\epsilon) - \frac{\epsilon}{t_\epsilon^2} - \frac{1}{2} \gamma \sigma^2 q_\epsilon^2 + V_{t_\epsilon} H(\partial_q \varphi(t_\epsilon, q_\epsilon)) \geq 0.$$

Hence:

$$-\partial_t \varphi(t_\epsilon, q_\epsilon) - \frac{1}{2} \gamma \sigma^2 q_\epsilon^2 + V_{t_\epsilon} H(\partial_q \varphi(t_\epsilon, q_\epsilon)) \geq 0,$$

and sending ϵ to 0 we obtain:

$$-\partial_t \varphi(0, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_0 H(\partial_q \varphi(0, q)) \geq 0.$$

Therefore, θ_T is a supersolution of the Hamilton-Jacobi equation on $[0, T) \times \mathbb{R}$. \square

We are now going to obtain convexity and semi-concavity properties for $\theta_T(\hat{t}, \cdot)$, $\forall \hat{t} \in [0, T)$.

Proposition 9 (Convexity of $\theta_T(t, \cdot)$). $\forall \hat{t} \in [0, T)$, $\theta_T(\hat{t}, \cdot)$ is a convex function.

Proof:

Let us consider $\hat{q} \in \mathbb{R}$ and $h > 0$.

We consider $q = \frac{1}{2} q_{\hat{t}, \hat{q}-h}^* + \frac{1}{2} q_{\hat{t}, \hat{q}+h}^*$ and we have:

$$\theta_T(\hat{t}, \hat{q} + h) - 2\theta_T(\hat{t}, \hat{q}) + \theta_T(\hat{t}, \hat{q} - h)$$

$$\begin{aligned}
&\geq \int_{\hat{t}}^T V_s \left(L \left(\frac{\dot{q}_{\hat{t}, \hat{q}+h}^*(s)}{V_s} \right) - 2L \left(\frac{\dot{q}(s)}{V_s} \right) + L \left(\frac{\dot{q}_{\hat{t}, \hat{q}-h}^*(s)}{V_s} \right) \right) ds \\
&\quad + \frac{1}{2} \gamma \sigma^2 \int_{\hat{t}}^T \left(q_{\hat{t}, \hat{q}+h}^*(s)^2 - 2q(s)^2 + q_{\hat{t}, \hat{q}-h}^*(s)^2 \right) ds.
\end{aligned}$$

Using the convexity of L and the convexity of $x \rightarrow x^2$, we obtain that:

$$\theta_T(\hat{t}, \hat{q} + h) - 2\theta_T(\hat{t}, \hat{q}) + \theta_T(\hat{t}, \hat{q} - h) \geq 0.$$

Hence, $\theta_T(\hat{t}, \cdot)$ is a convex function. \square

Proposition 10 (Local semi-concavity of $\theta_T(t, \cdot)$). *$\forall \hat{t} \in [0, T], \theta_T(\hat{t}, \cdot)$ is a locally semi-concave function.*

Proof:

Let us consider $Q > 0, \hat{q}_1 < \hat{q}_2 \in [-Q, Q]$ and $\lambda \in (0, 1)$.

Let us define $q_1 = q_{\hat{t}, \lambda \hat{q}_1 + (1-\lambda)\hat{q}_2}^* - (1-\lambda) \frac{T-\hat{t}}{T-\hat{t}} (\hat{q}_2 - \hat{q}_1)$ and $q_2 = q_{\hat{t}, \lambda \hat{q}_1 + (1-\lambda)\hat{q}_2}^* + \lambda \frac{T-\hat{t}}{T-\hat{t}} (\hat{q}_2 - \hat{q}_1)$. Then:

$$\begin{aligned}
&\lambda \theta_T(\hat{t}, \hat{q}_1) + (1-\lambda) \theta_T(\hat{t}, \hat{q}_2) - \theta_T(\hat{t}, \lambda \hat{q}_1 + (1-\lambda) \hat{q}_2) \\
&\leq \int_{\hat{t}}^T V_s \left(\lambda L \left(\frac{\dot{q}_1(s)}{V_s} \right) + (1-\lambda) L \left(\frac{\dot{q}_2(s)}{V_s} \right) - L \left(\frac{\dot{q}_{\hat{t}, \lambda \hat{q}_1 + (1-\lambda) \hat{q}_2}^*(s)}{V_s} \right) \right) ds \\
&\quad + \frac{1}{2} \gamma \sigma^2 \int_{\hat{t}}^T \left(\lambda q_1(s)^2 + (1-\lambda) q_2(s)^2 - q_{\hat{t}, \lambda \hat{q}_1 + (1-\lambda) \hat{q}_2}^*(s)^2 \right) ds.
\end{aligned}$$

Because L is C^1 , it is locally semi-concave and there exists a continuous and increasing function ω with $\omega(0) = 0$ such that (for $\epsilon = T - \hat{t}$):

$$\forall \rho_1, \rho_2 \in \left[-\frac{1}{\underline{V}} (C(Q, \epsilon) + 2\frac{Q}{\epsilon}), \frac{1}{\underline{V}} (C(Q, \epsilon) + 2\frac{Q}{\epsilon}) \right], \forall \lambda \in (0, 1),$$

$$\lambda L(\rho_1) + (1-\lambda) L(\rho_2) - L(\lambda \rho_1 + (1-\lambda) \rho_2) \leq \lambda(1-\lambda) |\rho_1 - \rho_2| \omega(|\rho_1 - \rho_2|).$$

Hence:

$$\begin{aligned}
&\lambda \theta_T(\hat{t}, \hat{q}_1) + (1-\lambda) \theta_T(\hat{t}, \hat{q}_2) - \theta_T(\hat{t}, \lambda \hat{q}_1 + (1-\lambda) \hat{q}_2) \\
&\leq \lambda(1-\lambda) \int_{\hat{t}}^T V_s \left| \frac{\dot{q}_2(s)}{V_s} - \frac{\dot{q}_1(s)}{V_s} \right| \omega \left(\left| \frac{\dot{q}_2(s)}{V_s} - \frac{\dot{q}_1(s)}{V_s} \right| \right) ds
\end{aligned}$$

$$+\frac{1}{2}\gamma\sigma^2\lambda(1-\lambda)(\hat{q}_2-\hat{q}_1)^2\int_{\hat{t}}^T\left(\frac{T-s}{T-\hat{t}}\right)^2ds.$$

Consequently:

$$\begin{aligned} & \lambda\theta_T(\hat{t}, \hat{q}_1) + (1-\lambda)\theta_T(\hat{t}, \hat{q}_2) - \theta_T(\hat{t}, \lambda\hat{q}_1 + (1-\lambda)\hat{q}_2) \\ & \leq \lambda(1-\lambda)|\hat{q}_2 - \hat{q}_1|\omega\left(\frac{|\hat{q}_2 - \hat{q}_1|}{\epsilon V}\right) + \frac{1}{6}\gamma\sigma^2\lambda(1-\lambda)\epsilon(\hat{q}_2 - \hat{q}_1)^2. \end{aligned}$$

This proves the semi-concavity of $\theta_T(\hat{t}, \cdot)$ on $[-Q, Q]$. This result being true for all Q , the local semi-concavity of $\theta_T(\hat{t}, \cdot)$ is proved. \square

Now, we prove an important result of this section stating that θ_T is in fact a C^1 function on $[0, T) \times \mathbb{R}$ solving the PDE of Proposition 8 in the classical sense.

Theorem 5 (Regularity of θ_T and Hamilton-Jacobi equation). $\theta_T \in C^1([0, T) \times \mathbb{R})$ and:

$$\forall (t, q) \in [0, T) \times \mathbb{R}, -\partial_t\theta_T(t, q) - \frac{1}{2}\gamma\sigma^2q^2 + V_tH(\partial_q\theta_T(t, q)) = 0.$$

Proof:

A first step in the proof consists in recalling that a function which is both locally semi-concave and convex is C^1 (see for instance Theorem 3.3.7 of [9]). This gives $\forall t \in [0, T), \theta_T(t, \cdot) \in C^1(\mathbb{R})$.

θ_T is C^0 on $[0, T) \times \mathbb{R}$ with $\forall t \in [0, T), \theta_T(t, \cdot)$ convex and C^1 . Hence, we deduce from Theorem 25.7 of [22] that $\partial_x\theta_T$ is in fact a continuous function on $[0, T) \times \mathbb{R}$.

Now, let us consider $q \in \mathbb{R}$. We know from Proposition 7 that $\theta_T(\cdot, q) \in W_{\text{loc}}^{1,\infty}(0, T)$, and then it is almost everywhere differentiable on $(0, T)$. If t is such that $\theta_T(\cdot, q)$ is differentiable at t , then θ_T is differentiable at (t, q) and we know that $-\partial_t\theta_T(t, q) - \frac{1}{2}\gamma\sigma^2q^2 + V_tH(\partial_q\theta_T(t, q)) = 0$. But this gives that almost everywhere on $[0, T)$, $\partial_t\theta_T(\cdot, q)$ is equal to a continuous function. Therefore $\theta_T(\cdot, q)$ is in fact $C^1([0, T))$ with $\partial_t\theta_T(t, q) = -\frac{1}{2}\gamma\sigma^2q^2 + V_tH(\partial_q\theta_T(t, q))$. We conclude that $\partial_t\theta_T$ is a continuous function. Therefore $\theta_T \in C^1([0, T) \times \mathbb{R})$ and θ_T solves the above PDE on $[0, T) \times \mathbb{R}$. \square

We proved that the function θ_T is a smooth function on $[0, T) \times \mathbb{R}$ and we exhibited the Hamilton-Jacobi equation solved by θ_T . We now go further in the study of θ_T and we start with monotonicity properties of θ_T .

Proposition 11. $\forall T > 0, \forall \hat{q} \in \mathbb{R}, \theta_T(\cdot, \hat{q})$ is an increasing function on $[0, T)$.

Proof:

Consider $t, \hat{t} \in [0, T)$ with $t < \hat{t}$. Let us define $q : s \in [t, T] \mapsto q_{\hat{t}, \hat{q}}^*(s - t + \hat{t})1_{s \leq T + t - \hat{t}}$.

We have:

$$\begin{aligned}
\theta_T(t, \hat{q}) &\leq \int_t^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q(s)^2 \right) ds \\
&\leq \int_t^{T+t-\hat{t}} \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q(s)^2 \right) ds \\
&\leq \int_{\hat{t}}^T \left(V_s L \left(\frac{\dot{q}_{\hat{t}, \hat{q}}^*(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q_{\hat{t}, \hat{q}}^*(s)^2 \right) ds \\
&\leq \theta_T(\hat{t}, \hat{q}).
\end{aligned}$$

Hence, $\theta_T(\cdot, \hat{q})$ is an increasing function on $[0, T)$. □

Corollary 4. *Consider $t \in \mathbb{R}_+$ and $q \in \mathbb{R}$. Then $T \in (t, +\infty) \mapsto \theta_T(t, q)$ is a decreasing function.*

Proof:

Consider $T, T' \in (t, +\infty)$ with $T < T'$. We have that $\theta_T(t, q) = \theta_{T'}(T' - T + t, q) \geq \theta_{T'}(t, q)$. □

These monotonicity results with respect to time will be useful to study the asymptotic behavior of the value function. We now turn to monotonicity results with respect to q .

Proposition 12. $\forall T > 0, \forall \hat{t} \in [0, T), q \in \mathbb{R}_+ \mapsto \theta_T(\hat{t}, q)$ is an increasing function.

Proof:

Consider $q, \hat{q} \in \mathbb{R}_+$ with $q < \hat{q}$. Then:

$$\begin{aligned}
\theta_T(\hat{t}, q) &\leq \int_t^T \left(V_s L \left(\frac{q \dot{q}_{t,\hat{q}}^*(s)}{\hat{q}} \right) + \frac{1}{2} \gamma \sigma^2 \left(\frac{q}{\hat{q}} \dot{q}_{t,\hat{q}}^*(s) \right)^2 \right) ds \\
&\leq \int_{\hat{t}}^T \left(V_s L \left(\frac{\dot{q}_{t,\hat{q}}^*(s)}{\hat{q}} \right) + \frac{1}{2} \gamma \sigma^2 \dot{q}_{t,\hat{q}}^*(s)^2 \right) ds \\
&\leq \theta_T(\hat{t}, \hat{q}).
\end{aligned}$$

Hence, $\theta_T(\hat{t}, \cdot)$ is an increasing function on \mathbb{R}_+ . \square

A natural corollary of these monotonicity results concerns the dependence of the liquidity premium on T and q :

Corollary 5. *For $q > 0$, the liquidity premium $qS - P(T, q, S) = \frac{k}{2}q^2 + \psi q + \theta_T(0, q)$ is an increasing function of q and a decreasing function of T .*

Regarding the price of a block trade, in addition to the above natural monotonicity properties, two questions arise:

- Since an agent buying a portfolio with q shares may not be time-constrained to liquidate it, we have to study the limit of $P(T, q, S)$ as $T \rightarrow +\infty$ – this limit being then an upper bound for any other price with T finite.
- Since θ_T has a singularity at time $t = T$, it cannot be approximated using a numerical scheme on the Hamilton-Jacobi PDE solved by θ_T . At least not directly. The question of the numerical approximation of θ_T and $P(T, q, S)$ requires to approximate θ_T by another function without singularity.

4.2 Block trade pricing in the time-unconstrained case

So far, we only defined the price of a block trade when the agent buying the portfolio was constrained to liquidate it on the market within a certain time T . It is also interesting to focus on the time-unconstrained case. To that purpose, one needs to study the asymptotic behavior of $P(T, q, S)$ as $T \rightarrow +\infty$, or equivalently the asymptotic behavior of $T \mapsto \theta_T$.

Theorem 6 (Asymptotic behavior of θ_T). *Let us consider the case of a constant market volume curve $V_t = V$.*

$$\forall q \geq 0, \forall t \geq 0, \lim_{T \rightarrow +\infty} \theta_T(t, q) = \theta_\infty(q) = \int_0^q H^{-1} \left(\frac{\gamma \sigma^2}{2V} x^2 \right) dx,$$

where H^{-1} is the inverse of $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Proof:

For $(t, q) \in \mathbb{R}_+ \times \mathbb{R}_+$, $T > t \mapsto \theta_T(t, q)$ is a decreasing function bounded from below by 0. Hence $\lim_{T \rightarrow +\infty} \theta_T(t, q)$ exists.

Since for $0 \leq t' < t$, $\theta_T(t', q) = \theta_{T-t'+t}(t, q)$, $\lim_{T \rightarrow +\infty} \theta_T(t, q)$ is in fact independent of t and we define $\theta_\infty(q) = \lim_{T \rightarrow +\infty} \theta_T(t, q)$.

Using Proposition 9, θ_∞ is a convex function and it is therefore continuous.

Hence, for a fixed $t \geq 0$, using Dini's theorem, the convergence of $(\theta_T(t, \cdot))_T$ towards θ_∞ is locally uniform on \mathbb{R}_+ . Now, because of Proposition 11, we have that $(\theta_T)_T$ converges towards θ_∞ locally uniformly on $\mathbb{R}_+ \times \mathbb{R}_+$.

Now, we are going to use the Hamilton-Jacobi equation of Proposition 8 to prove that θ_∞ is a viscosity solution of:

$$-\frac{1}{2}\gamma\sigma^2q^2 + VH(\theta'_\infty(q)) = 0, \quad q > 0, \quad \theta_\infty(0) = 0 \quad (*).$$

We consider $q > 0$ and $\varphi \in C^1(\mathbb{R}_+^*)$ such that $\theta_\infty - \varphi$ has a local strict maximum in q . Then, we consider a sequence of triplets $(T_n, t_n, q_n) \in \mathbb{R}_+^2 \times \mathbb{R}_+^*$ with $t_n < T_n$ such that $\lim_n T_n = +\infty$, $(t_n, q_n)_n$ converges with $\lim_n q_n = q$, and $\theta_{T_n} - \varphi$ has a local maximum in (t_n, q_n) . We therefore obtain that:

$$-\frac{1}{2}\gamma\sigma^2q_n^2 + VH(\varphi'(q_n)) \leq 0.$$

Hence, considering the limit $n \rightarrow +\infty$, we have:

$$-\frac{1}{2}\gamma\sigma^2q^2 + VH(\varphi'(q)) \leq 0.$$

Conversely, if we consider $q > 0$ and $\varphi \in C^1(\mathbb{R}_+^*)$ such that $\theta_\infty - \varphi$ has a local minimum in q . Then, we consider a sequence of triplets $(T_n, t_n, q_n) \in \mathbb{R}_+^2 \times \mathbb{R}_+^*$ with $t_n < T_n$ such that $\lim_n T_n = +\infty$, $(t_n, q_n)_n$ converges with $\lim_n q_n = q$, and $\theta_{T_n} - \varphi$ has a local minimum in (t_n, q_n) . We therefore obtain that:

$$-\frac{1}{2}\gamma\sigma^2q_n^2 + VH(\varphi'(q_n)) \geq 0.$$

Hence, considering the limit $n \rightarrow +\infty$, we have:

$$-\frac{1}{2}\gamma\sigma^2q^2 + VH(\varphi'(q)) \geq 0.$$

This, and the fact that $\theta_\infty(0) = 0$, proves that θ_∞ is indeed a viscosity solution of

the equation (*).

Now, θ_∞ is convex and hence we have almost everywhere that $-\frac{1}{2}\gamma\sigma^2q^2 + VH(\theta'_\infty(q)) = 0$. Since θ_∞ is an increasing function (as a limit of such functions), we can write that almost everywhere:

$$\theta'_\infty(q) = H^{-1}\left(\frac{1}{2V}\gamma\sigma^2q^2\right),$$

where H^{-1} is the inverse of $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, as defined in the above statement.

Since θ_∞ is convex we can integrate this equation to obtain:

$$\theta_\infty(q) = \theta_\infty(0) + \int_0^q H^{-1}\left(\frac{\gamma\sigma^2}{2V}x^2\right) dx = \int_0^q H^{-1}\left(\frac{\gamma\sigma^2}{2V}x^2\right) dx,$$

and the result is proved. \square

A straightforward consequence of this result is the following Theorem that gives the price of a block trade in closed-form in the time-unconstrained case:

Theorem 7 (Closed-form expression). *Let us consider the case of a constant market volume curve $V_t = V$.*

The price of a block trade for a portfolio with $q > 0$ shares is:

$$P(q, S) := \lim_{T \rightarrow +\infty} P(T, q, S) = qS - \frac{k}{2}q^2 - \psi q - \int_0^q H^{-1}\left(\frac{\gamma\sigma^2}{2V}x^2\right) dx,$$

where H^{-1} is the inverse of $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

As an illustration, we can compute the price of a block trade when the execution cost function L is $L(\rho) = \eta|\rho|^{1+\phi}$:

Proposition 13 (Block trade pricing for $L(\rho) = \eta|\rho|^{1+\phi}$). *Let us suppose that $V_t = V$ and that $L(\rho) = \eta|\rho|^{1+\phi}$.*

Then:

$$P(q, S) = qS - \frac{k}{2}q^2 - \psi q - \frac{\eta^{\frac{1}{1+\phi}}(1+\phi)^2}{\phi^{\frac{\phi}{1+\phi}}1+3\phi} \left(\frac{\gamma\sigma^2}{2V}\right)^{\frac{\phi}{1+\phi}} q^{\frac{1+3\phi}{1+\phi}}.$$

Proof:

When $L(\rho) = \eta|\rho|^{1+\phi}$, the function H is given by $H(p) = \frac{\phi}{(1+\phi)^{1+\frac{1}{\phi}}} \frac{1}{\eta^{\frac{1}{\phi}}} |p|^{1+\frac{1}{\phi}}$. Hence,

for $x \geq 0$, $H^{-1}(x) = \eta^{\frac{1}{1+\phi}} \frac{1+\phi}{\phi^{\frac{\phi}{1+\phi}}} x^{\frac{\phi}{1+\phi}}$.

Using now the result of Theorem 7, we obtain the formula. \square

4.3 Approximation of θ_T

In order to approximate the function θ_T , we are going to consider a control problem slightly different from the initial one in which the end point q_T was constrained to be equal to 0. Now, we consider trajectories for the inventory that do not necessarily correspond to liquidation strategies. However, we penalize large inventories at the final time T .

Given $K > 0$, we define for $(\hat{t}, \hat{q}) \in [0, T] \times \mathbb{R}$:

$$\theta_{K,T}(\hat{t}, \hat{q}) = \inf_{q \in W_{\hat{q}}^{1,1+\epsilon}(\hat{t}, T)} \int_{\hat{t}}^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q^2(s) \right) ds + K q(T)^2.$$

where $W_a^{1,1+\epsilon}(t_1, t_2)$ is the set of functions $q \in W^{1,1+\epsilon}(t_1, t_2)$ satisfying $q(t_1) = a$.

Using the same ideas as above (or sometimes even more classical methods since there is no singularity at time $t = T$), we can obtain the following theorem:

Theorem 8 (The relaxed problem). *Let us fix $K > 0$. $\forall (\hat{t}, \hat{q}) \in [0, T] \times \mathbb{R}$, the infimum that defines $\theta_{K,T}(\hat{t}, \hat{q})$ is attained by a unique function $q_{K,\hat{t},\hat{q}}^* \in W_{\hat{q}}^{1,1+\epsilon}(\hat{t}, T)$ that is in fact in $C^1([0, T])$.*

Coming to the value function, $\theta_{K,T}$ is a $C^1([0, T] \times \mathbb{R})$ function, solution of the PDE:

$$\forall (t, q) \in [0, T] \times \mathbb{R}, -\partial_t \theta_{K,T}(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H(\partial_q \theta_{K,T}(t, q)) = 0,$$

with terminal condition $\theta_{K,T}(T, q) = K q^2$ and “boundary” condition $\theta_{K,T}(t, 0) = 0$.

To obtain a differential characterization of $\theta_{K,T}$, the Hamilton-Jacobi equation can be written in another form that guarantees uniqueness:

Proposition 14 (Another Hamilton-Jacobi equation). *Consider $Q > 0$. $\theta_{K,T}$ is the unique continuous viscosity solution of:*

$$-\partial_t \theta_{K,T}(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H((\partial_q \theta_{K,T}(t, q))_+) = 0, \quad \text{on } [0, T] \times (0, Q].$$

$$\theta_{K,T}(t, 0) = 0, \forall t \in [0, T], \quad \theta_{K,T}(T, q) = K q^2, \forall q \in [0, Q].$$

Proof:

Let $(t, q) \in [0, T] \times (0, Q]$ and $\varphi \in C^1([0, T] \times (0, Q])$ be such that $\theta_{K,T} - \varphi$ has a local maximum on $[0, T] \times (0, Q]$ in (t, q) . Then:

$$\partial_t \theta_{K,T}(t, q) - \partial_t \varphi(t, q) \leq 0, \quad \partial_q \theta_{K,T}(t, q) - \partial_q \varphi(t, q) \geq 0.$$

Now, since $p \mapsto H(p_+)$ is increasing, we have:

$$\begin{aligned}
& -\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H((\partial_q \varphi(t, q))_+) \\
& \leq -\partial_t \theta_{K,T}(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H((\partial_q \theta_{K,T}(t, q))_+) = 0.
\end{aligned}$$

This proves the subsolution part.

Let now $(t, q) \in [0, T) \times (0, Q]$ and $\varphi \in C^1([0, T) \times (0, Q])$ be such that $\theta_{K,T} - \varphi$ has a local minimum on $[0, T) \times (0, Q]$ in (t, q) . Then:

$$\partial_t \theta_{K,T}(t, q) - \partial_t \varphi(t, q) \geq 0, \quad \partial_q \theta_{K,T}(t, q) - \partial_q \varphi(t, q) \leq 0.$$

Now, since $p \mapsto H(p_+)$ is increasing, we have:

$$\begin{aligned}
& -\partial_t \varphi(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H((\partial_q \varphi(t, q))_+) \\
& \geq -\partial_t \theta_{K,T}(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + V_t H((\partial_q \theta_{K,T}(t, q))_+) = 0.
\end{aligned}$$

This proves the supersolution part.

Since $\theta_{K,T}(t, 0) = 0, \forall t \in [0, T]$ and $\theta_{K,T}(T, q) = Kq^2, \forall q \in [0, Q]$, $\theta_{K,T}$ is indeed a viscosity solution of the equation, as announced.

Now, to prove that there is a unique continuous viscosity solution of the equation we use the classical *doubling of variables* method.

Let us consider two continuous viscosity solutions of the equation: θ^1 and θ^2 . We consider $a > 0$ and for $\epsilon > 0, \nu > 0$,

$$\begin{aligned}
U^{\epsilon, \nu} : [0, T]^2 \times [0, Q]^2 & \rightarrow \mathbb{R} \\
(t_1, t_2, q_1, q_2) & \mapsto \theta^1(t_1, q_1) - \theta^2(t_2, q_2) - \frac{(t_1 - t_2)^2}{2\epsilon} - \frac{(q_1 - q_2)^2}{2\nu} - a(t_1 + t_2).
\end{aligned}$$

$\forall \epsilon > 0, \forall \nu > 0$, we consider $(t_1^{\epsilon, \nu}, t_2^{\epsilon, \nu}, q_1^{\epsilon, \nu}, q_2^{\epsilon, \nu}) \in [0, T]^2 \times [0, Q]^2$ such that the maximum of $U^{\epsilon, \nu}$ is attained in $(t_1^{\epsilon, \nu}, t_2^{\epsilon, \nu}, q_1^{\epsilon, \nu}, q_2^{\epsilon, \nu})$.

We have $U^{\epsilon, \nu}(t_1^{\epsilon, \nu}, t_1^{\epsilon, \nu}, q_1^{\epsilon, \nu}, q_1^{\epsilon, \nu}) \leq U^{\epsilon, \nu}(t_1^{\epsilon, \nu}, t_2^{\epsilon, \nu}, q_1^{\epsilon, \nu}, q_2^{\epsilon, \nu})$. Hence:

$$\frac{1}{2\nu}(q_1^{\epsilon, \nu} - q_2^{\epsilon, \nu})^2 + \frac{1}{2\epsilon}(t_1^{\epsilon, \nu} - t_2^{\epsilon, \nu})^2 \leq \theta^2(t_1^{\epsilon, \nu}, q_1^{\epsilon, \nu}) - \theta^2(t_2^{\epsilon, \nu}, q_2^{\epsilon, \nu}) - a(t_2^{\epsilon, \nu} - t_1^{\epsilon, \nu}).$$

This gives that $t_1^{\epsilon, \nu} - t_2^{\epsilon, \nu} = \mathcal{O}(\sqrt{\epsilon})$, uniformly in ν , and $q_1^{\epsilon, \nu} - q_2^{\epsilon, \nu} = \mathcal{O}(\sqrt{\nu})$, uniformly in ϵ .

Let us define $B = (\{T\} \times [0, Q]) \cup ([0, T] \times \{0\})$. We have to distinguish two cases:

- Case (i): There exists a sequence $(\epsilon_n, \nu_n)_n$ converging towards $(0, 0)$ such that $\forall n \in \mathbb{N}$, $(t_1^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}) \in B$ or $(t_2^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}) \in B$.

If $(t_1^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}) \in B$ then:

$$\begin{aligned} & U^{\epsilon_n, \nu_n}(t_1^{\epsilon_n, \nu_n}, t_2^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}) \\ & \leq \theta^1(t_1^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}) - \theta^2(t_2^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}) = \theta^2(t_1^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}) - \theta^2(t_2^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}). \end{aligned}$$

Similarly, if $(t_2^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}) \in B$ then:

$$\begin{aligned} & U^{\epsilon_n, \nu_n}(t_1^{\epsilon_n, \nu_n}, t_2^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}) \\ & \leq \theta^1(t_1^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}) - \theta^2(t_2^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}) = \theta^1(t_1^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}) - \theta^1(t_2^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}). \end{aligned}$$

We conclude that $\liminf_{n \rightarrow +\infty} U^{\epsilon_n, \nu_n}(t_1^{\epsilon_n, \nu_n}, t_2^{\epsilon_n, \nu_n}, q_1^{\epsilon_n, \nu_n}, q_2^{\epsilon_n, \nu_n}) \leq 0$.

- Case (ii): For ϵ and ν below a certain threshold, we always have $(t_1^{\epsilon, \nu}, q_1^{\epsilon, \nu}) \notin B$ and $(t_2^{\epsilon, \nu}, q_2^{\epsilon, \nu}) \notin B$.

In that case, we use the fact the θ^1 is a subsolution to obtain:

$$a + \frac{1}{\epsilon}(t_1^{\epsilon, \nu} - t_2^{\epsilon, \nu}) - \frac{1}{2}\gamma\sigma^2 q_1^{\epsilon, \nu 2} + V_{t_1^{\epsilon, \nu}} H \left(\left(\frac{1}{\nu}(q_1^{\epsilon, \nu} - q_2^{\epsilon, \nu}) \right)_+ \right) \leq 0.$$

Similarly, we use the fact the θ^2 is a supersolution to obtain:

$$-a + \frac{1}{\epsilon}(t_1^{\epsilon, \nu} - t_2^{\epsilon, \nu}) - \frac{1}{2}\gamma\sigma^2 q_2^{\epsilon, \nu 2} + V_{t_2^{\epsilon, \nu}} H \left(\left(\frac{1}{\nu}(q_1^{\epsilon, \nu} - q_2^{\epsilon, \nu}) \right)_+ \right) \geq 0.$$

Hence,

$$2a - \frac{1}{2}\gamma\sigma^2 (q_1^{\epsilon, \nu 2} - q_2^{\epsilon, \nu 2}) + (V_{t_1^{\epsilon, \nu}} - V_{t_2^{\epsilon, \nu}}) H \left(\left(\frac{1}{\nu}(q_1^{\epsilon, \nu} - q_2^{\epsilon, \nu}) \right)_+ \right) \leq 0.$$

Sending ϵ to 0 and then sending ν to 0, we get that $2a \leq 0$. This is not possible.

We then conclude that

$$\sup_{(t, q) \in [0, T] \times [0, Q]} \theta^1(t, q) - \theta^2(t, q) - 2at \leq \liminf_{\epsilon, \nu \rightarrow 0} U^{\epsilon, \nu}(t_1^{\epsilon, \nu}, t_1^{\epsilon, \nu}, q_1^{\epsilon, \nu}, q_1^{\epsilon, \nu}) \leq 0.$$

Thus $\theta^1 - \theta^2 \leq 2aT$ and sending a to 0, we get $\theta^1 \leq \theta^2$

Reversing the role played by θ^1 and θ^2 , we eventually obtain $\theta^1 = \theta^2$.

Therefore, there is a unique continuous viscosity solution and the result is proved. \square

The above characterization is useful to obtain numerical schemes but the main result of this subsection is the convergence of $\theta_{K,T}$ toward θ_T . This convergence result which is stated in the next theorem will allow to approximate θ_T numerically since $\theta_{K,T}$ can be computed numerically using a monotone scheme for the above Hamilton-Jacobi equation.

Theorem 9 (Convergence of $\theta_{K,T}$).

$$\forall (t, q) \in [0, T) \times \mathbb{R}, \lim_{K \rightarrow +\infty} \theta_{K,T}(t, q) = \theta_T(t, q).$$

Proof:

Let us consider $(\hat{t}, \hat{q}) \in [0, T) \times \mathbb{R}$. By definition, $\theta_{K,T}(\hat{t}, \hat{q}) \leq \theta_T(\hat{t}, \hat{q})$.

In particular, $Kq_{K,\hat{t},\hat{q}}^*(T)^2 \leq \theta_T(\hat{t}, \hat{q})$, and this leads to $\lim_{K \rightarrow +\infty} q_{K,\hat{t},\hat{q}}^*(T) = 0$.

Now,

$$\int_{\hat{t}}^T \left(A \frac{|\dot{q}_{K,\hat{t},\hat{q}}^*(s)|^{1+\epsilon}}{V_s^\epsilon} - BV_s \right) ds \leq \int_{\hat{t}}^T V_s L \left(\frac{\dot{q}_{K,\hat{t},\hat{q}}^*(s)}{V_s} \right) ds \leq \theta_{K,T}(\hat{t}, \hat{q}) \leq \theta_T(\hat{t}, \hat{q}).$$

Hence $\sup_{K \in \mathbb{N}} \|\dot{q}_{K,\hat{t},\hat{q}}^*\|_{L^{1+\epsilon}(\hat{t}, T)} < +\infty$. Now, since $|q_{K,\hat{t},\hat{q}}^*| \leq \hat{q}$, we know that $(q_{K,\hat{t},\hat{q}}^*)_{K \in \mathbb{N}}$ is a bounded sequence of $W_{\hat{q}}^{1,1+\epsilon}(\hat{t}, T)$. Hence, there exists $q \in W_{\hat{q}}^{1,1+\epsilon}(\hat{t}, T)$ such that, up to a subsequence, $q_{K,\hat{t},\hat{q}}^* \rightarrow q$ in $C^0([0, T])$, $q(0) = 0$, and $\dot{q}_{K,\hat{t},\hat{q}}^* \rightharpoonup \dot{q}$ in $L^{1+\epsilon}(\hat{t}, T)$.

Now, using the convexity of L as in the proof of Theorem 1, we have that:

$$\begin{aligned} \theta_T(\hat{t}, \hat{q}) &\leq \int_0^T \left(V_s L \left(\frac{\dot{q}(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q^2(s) \right) ds \\ &\leq \liminf_{K \rightarrow +\infty} \int_0^T V_s L \left(\frac{\dot{q}_{K,\hat{t},\hat{q}}^*(s)}{V_s} \right) ds + \frac{1}{2} \gamma \sigma^2 \int_0^T q(s)^2 ds \\ &\leq \liminf_{K \rightarrow +\infty} \int_0^T V_s L \left(\frac{\dot{q}_{K,\hat{t},\hat{q}}^*(s)}{V_s} \right) ds + \frac{1}{2} \gamma \sigma^2 \lim_{K \rightarrow +\infty} \int_0^T q_{K,\hat{t},\hat{q}}^*(s)^2 ds \\ &\leq \liminf_{K \rightarrow +\infty} \int_0^T V_s L \left(\frac{\dot{q}_{K,\hat{t},\hat{q}}^*(s)}{V_s} \right) ds + \frac{1}{2} \gamma \sigma^2 \int_0^T q_{K,\hat{t},\hat{q}}^*(s)^2 ds \\ &\leq \liminf_{K \rightarrow +\infty} \theta_{K,T}(\hat{t}, \hat{q}) \end{aligned}$$

Now, since $\theta_{K,T}(\hat{t}, \hat{q})$ is increasing in K , we indeed have that $\lim_{K \rightarrow +\infty} \theta_{K,T}(t, q) = \theta_T(t, q)$. \square

5 Limitation to deterministic strategies: another route

In Section 3, we proved that the deterministic strategy v^* is in fact optimal in a wider class of stochastic strategies. The purpose of this section is to provide another proof of this result using a relaxed problem in which liquidation is not enforced.

To start with, we introduce the set of admissible strategies of our relaxed problem:

$$\mathcal{A}(t, q) = \left\{ (v_s)_{s \in [t, T]} \in \mathcal{P}(t, T), (s, \omega) \mapsto \int_t^s v_\tau(\omega) d\tau \in L^\infty([t, T] \times \Omega) \right\}.$$

Similarly, let us define the set of admissible liquidation strategies:

$$\mathcal{A}_0(t, q) = \left\{ (v_s)_{s \in [t, T]} \in \mathcal{A}(t, q), \int_t^T v_s ds = q \right\}.$$

Proposition 15 (Verification theorem for the relaxed problem). *Let us consider $K > 0$.*

Let us define for $(t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$:

$$u_K(t, x, q, S) = -\exp \left(-\gamma \left(x + qS - \frac{k}{2}q^2 - \psi q - \theta_{K,T}(t, q) \right) \right).$$

Then

$$u_K(t, x, q, S) = \sup_{v \in \mathcal{A}(t, q)} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t, x, v} + q_T^{t, q, v} S_T^{t, S, v} - \frac{k}{2} q_T^{t, q, v^2} - \psi q_T^{t, q, v} - K q_T^{t, q, v^2} \right) \right) \right],$$

where $\forall \tau \in [t, T]$:

$$\begin{aligned} q_\tau^{t, q, v} &= q - \int_t^\tau v_s ds, \\ S_\tau^{t, S, v} &= S + \int_t^\tau \sigma dW_s - k v_s ds, \\ X_\tau^{t, x, v} &= x + \int_t^\tau \left(v_s S_s^{t, S, v} - V_s L \left(\frac{v_s}{V_s} \right) - \psi v_s \right) ds. \end{aligned}$$

Proof:

Let us consider $v \in \mathcal{A}(t, q)$ such that $\int_t^T V_s L\left(\frac{v_s}{V_s}\right) ds < +\infty$. Using Itô's Lemma, and removing the superscripts to ease readability, we have:

$$u_K(T, X_T, q_T, S_T)$$

$$\begin{aligned} &= u_K(t, x, q, S) + \int_t^T \partial_t u_K(s, X_s, q_s, S_s) ds + \frac{1}{2} \sigma^2 \int_t^T \partial_{SS}^2 u_K(s, X_s, q_s, S_s) ds \\ &+ \int_t^T \left(v_s S_s - V_s L\left(\frac{v_s}{V_s}\right) - \psi v_s \right) \partial_x u_K(s, X_s, q_s, S_s) ds - \int_t^T v_s \partial_q u_K(s, X_s, q_s, S_s) ds \\ &- \int_t^T k v_s \partial_S u_K(s, X_s, q_s, S_s) ds + \int_t^T \partial_S u_K(s, X_s, q_s, S_s) \sigma dW_s \end{aligned}$$

Hence, using the expression for u_K , we get:

$$\begin{aligned} u_K(T, X_T, q_T, S_T) &= u_K(t, x, q, S) + \int_t^T \gamma u_K(s, X_s, q_s, S_s) \left(\partial_t \theta_{K,T}(s, q_s) + \frac{1}{2} \gamma \sigma^2 q_s^2 \right) ds \\ &- \int_t^T \gamma u_K(s, X_s, q_s, S_s) \left(v_s S_s - V_s L\left(\frac{v_s}{V_s}\right) - \psi |v_s| - v_s (S_s - k q_s - \psi - \partial_q \theta_{K,T}(s, q_s)) \right) ds \\ &+ \int_t^T \gamma k v_s q_s u_K(s, X_s, q_s, S_s) ds - \int_t^T \gamma \sigma q_s u_K(s, X_s, q_s, S_s) dW_s. \end{aligned}$$

Simplifying this expression, and using the fact that $-\psi(|v| - v) \leq 0$, we get:

$$\begin{aligned} u_K(T, X_T, q_T, S_T) &\leq u_K(t, x, q, S) - \int_t^T \gamma \sigma q_s u_K(s, X_s, q_s, S_s) dW_s, \\ &- \int_t^T \gamma u_K(s, X_s, q_s, S_s) \left(-\partial_t \theta_{K,T}(s, q_s) - \frac{1}{2} \gamma \sigma^2 q_s^2 + \partial_q \theta_{K,T}(s, q_s) v_s - V_s L\left(\frac{v_s}{V_s}\right) \right) ds. \end{aligned}$$

Using the PDE solved by $\theta_{K,T}$, we obtain that

$$-\partial_t \theta_{K,T}(s, q_s) - \frac{1}{2} \gamma \sigma^2 q_s^2 + \partial_q \theta_{K,T}(s, q_s) v_s - V_s L\left(\frac{v_s}{V_s}\right) \leq 0.$$

Therefore:

$$u_K(T, X_T, q_T, S_T) \leq u_K(t, x, q, S) - \int_t^T \gamma \sigma q_s u_K(s, X_s, q_s, S_s) dW_s.$$

Now, the definition of $\mathcal{A}(t, q)$ guarantees that $(q_s)_{s \in [t, T]}$ is bounded and hence that the local martingale $\int_t^T \gamma \sigma q_s u_K(s, X_s, q_s, S_s) dW_s$ is in fact a martingale. Consequently:

$$\mathbb{E}[u_K(T, X_T, q_T, S_T)] \leq u_K(t, x, q, S),$$

i.e.:

$$\mathbb{E} \left[-\exp \left(-\gamma \left(X_T + q_T S_T - \frac{k}{2} q_T^2 - \psi q_T - K q_T^2 \right) \right) \right] \leq u_K(t, x, q, S).$$

Now

$$\begin{aligned} u_K(t, x, q, S) &= -\exp \left(-\gamma \left(x + qS - \frac{k}{2} q^2 - \psi q - \theta_{K,T}(t, q) \right) \right) \\ &= -\exp \left(-\gamma \left(x + qS - \frac{k}{2} q^2 - \psi q \right. \right. \\ &\quad \left. \left. - \int_t^T \left(V_s L \left(\frac{\dot{q}_{K,t,q}^*(s)}{V_s} \right) + \frac{1}{2} \gamma \sigma^2 q_{K,t,q}^*(s)^2 \right) ds - K q_{K,t,q}^*(T)^2 \right) \right) \\ &= \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^* + q_T^* S_T^* - \frac{k}{2} q_T^{*2} - \psi q_T^* - K q_T^{*2} \right) \right) \right], \end{aligned}$$

where (X^*, q^*, S^*) corresponds to the admissible control $v_K^* = -\dot{q}_{K,t,q}^*$.

Combining the results, we obtain:

$$u_K(t, x, q, S) = \sup_{v \in \mathcal{A}(t,q)} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T + q_T S_T - \frac{k}{2} q_T^2 - \psi q_T - K q_T^2 \right) \right) \right].$$

□

In spite of the singularity associated to the liquidation problem, we now state a result that gives the value function of our initial stochastic optimal control problem and proves that the optimal deterministic strategy of Theorem 2 is in fact optimal in the wider set of controls \mathcal{A}_0 .

Theorem 10 (Verification theorem and optimality of deterministic strategies). *Let us define for $(t, x, q, S) \in [0, T) \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$:*

$$\begin{aligned} u(t, x, q, S) &= \lim_{K \rightarrow \infty} u_K(t, x, q, S) \\ &= -\exp \left(-\gamma \left(x + qS - \frac{k}{2} q^2 - \psi q - \theta_T(t, q) \right) \right). \end{aligned}$$

Then:

$$\begin{aligned} u(t, x, q, S) &= \sup_{v \in \mathcal{A}_0(t,q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t,x,v} \right) \right] \\ &= \sup_{v \in \mathcal{A}_{0,\det}(t,q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t,x,v} \right) \right] \end{aligned}$$

and this supremum is reached by the optimal deterministic strategy $v^* = -\dot{q}_{t,q}^*$.

Proof:

By definition we have:

$$\begin{aligned} & \sup_{v \in \mathcal{A}_0(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right] \\ & \leq \sup_{v \in \mathcal{A}(t, q)} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t, x, v} + q_T^{t, q, v} S_T^{t, S, v} - \frac{k}{2} q_T^{t, q, v^2} - \psi q_T^{t, q, v} - K q_T^{t, q, v^2} \right) \right) \right]. \end{aligned}$$

Hence:

$$\sup_{v \in \mathcal{A}_0(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right] \leq u_K(t, x, q, S).$$

Passing to the limit as $K \rightarrow \infty$ and using Theorem 9, we obtain that

$$\sup_{v \in \mathcal{A}_0(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right] \leq u(t, x, q, S).$$

As above, a straightforward computation indicates then that $u(t, x, q, S)$ is nothing but the expected value $\mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v^*} \right) \right]$ where $v^* = -\dot{q}_{t, q}^* \in \mathcal{A}_{0, \det}$.

Hence

$$u(t, x, q, S) \leq \sup_{v \in \mathcal{A}_{0, \det}(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right].$$

Since we obviously have

$$\sup_{v \in \mathcal{A}_{0, \det}(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right] \leq \sup_{v \in \mathcal{A}_0(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right],$$

we must have:

$$\begin{aligned} u(t, x, q, S) &= \sup_{v \in \mathcal{A}_0(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right] \\ &= \sup_{v \in \mathcal{A}_{0, \det}(t, q)} \mathbb{E} \left[-\exp \left(-\gamma X_T^{t, x, v} \right) \right], \end{aligned}$$

and the optimal strategy is indeed $v^* = -\dot{q}_{t, q}^*$. □

6 Numerical methods

In this section we discuss the numerical methods that may be used to compute the optimal liquidation strategy and/or an approximation of the price of a block trade. The goal of this section is not to prove any convergence result but rather to warn

the reader about potential problems when using numerical methods to approximate the solutions of our problems.

6.1 The flat market volume curve case

Let us first start with the simplest case where the market volume curve is flat ($V_t = V$). We proved in Theorem 3 that the optimal liquidation strategy was the solution of a first order ODE with maybe an unknown constant. If we use the notations of Theorem 3, we know that there are two cases:

- If $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is not integrable in $\xi = 0$ then there exists $C > 0$ such that q^* is uniquely characterized by:

$$q^*(t) = -Vr^{-1} \left(\frac{\gamma\sigma^2}{2V} q^*(t)^2 + C \right) \quad \text{with} \quad q^*(T) = 0$$

A consequence of this result is that q^* can be approximated using a finite difference scheme and a shooting method. More precisely, if we consider a grid $t_0 = 0 < t_1 < \dots < t_n = n\Delta t < \dots < t_N = T$, we can introduce for a given $C > 0$ the sequence $(q_n^C)_n$ defined by:

$$q_N^C = 0, \quad \forall n \leq N, q_{n-1}^C = q_n^C + V\Delta tr^{-1} \left(\frac{\gamma\sigma^2}{2V} q_n^{C^2} + C \right)$$

Then, we easily see that q_0^C is a continuous and strictly increasing function of C and we can therefore easily find C numerically such that $q_0^C = q_0$.

- If $\xi \mapsto \frac{1}{r^{-1}(\xi^2)}$ is integrable in $\xi = 0$, then the method depends on q_0 . If q_0 is strictly larger than q_0^{\lim} defined by:

$$\begin{cases} q_0^{\lim} = +\infty, & \text{if } \int_0^\infty \frac{1}{Vr^{-1}\left(\frac{\gamma\sigma^2}{2V}\xi^2\right)} d\xi < T, \\ \int_0^{q_0^{\lim}} \frac{1}{Vr^{-1}\left(\frac{\gamma\sigma^2}{2V}\xi^2\right)} d\xi = T, & \text{otherwise} \end{cases},$$

the same shooting method as above can be used.

On the contrary, if q_0 is lesser than q_0^{\lim} , we consider the following scheme:

$$q_0 = q_0, \quad \forall n \geq 0, q_{n+1} = \left[q_n - V\Delta tr^{-1} \left(\frac{\gamma\sigma^2}{2V} q_n^2 \right) \right]_+.$$

This method works very well in practice because in the case of a flat market volume curve we exactly know when to consider a forward scheme or a backward scheme. In the case of a non-constant volume curve, there is no such result and in any case the problem does not boil down to a first order ODE.

6.2 The Hamiltonian system

In the general case of a non-constant market volume curve, we exhibited two differential characterizations, namely the Euler-Lagrange equation and the Hamiltonian system of equations. We do prefer to use the Hamiltonian system to approximate numerically the solution.

Let us recall that q^* is solution of

$$\begin{cases} \dot{p}(t) &= \gamma\sigma^2 q(t) \\ \dot{q}(t) &= V_t H'(p(t)) \end{cases} \quad q(0) = q_0, \quad q(T) = 0$$

In order to compute q^* numerically, we consider the backward Cauchy problem:

$$\begin{cases} \dot{p}(t) &= \gamma\sigma^2 q(t) \\ \dot{q}(t) &= V_t H'(p(t)) \end{cases} \quad p(T) = p_T \quad q(T) = 0$$

where p_T is a (negative) constant and we want to find p_T such that $q(0) = q_0$.

Numerically this leads to the following scheme:

$$\begin{cases} \forall n \leq N, p_{n-1}^\lambda &= p_n^\lambda - \gamma\sigma^2 \Delta t q_n^\lambda \\ \forall n \leq N, q_{n-1}^\lambda &= q_n^\lambda - V_n \Delta t H'(p_n^\lambda) \end{cases} \quad p_N^\lambda = \lambda \quad q_N^\lambda = 0$$

One can easily show that for $q_0 > 0$ there exists a unique $\lambda < 0$ such that $q_0^\lambda = q_0$ (the function $\lambda < 0 \mapsto q_0^\lambda$ is continuous and strictly decreasing). However, finding such a λ is sometimes complicated. The reason for that can be well understood in the case of a flat market volume curve. We know indeed in that case that some optimal trajectories reach 0 before time T (this is the reason why we used a forward equation when q_0 was small in the flat market volume curve case) and therefore that $p(T) = 0$. Hence λ must be really small to obtain $q_0^\lambda = q_0$.

When the appropriate λ seems to be unreachable numerically, a solution can be to consider the following forward scheme:

$$\begin{cases} \forall n \leq N, p_{n+1}^\mu &= p_n^\mu + \gamma\sigma^2 \Delta t q_n^\mu \\ \forall n \leq N, q_{n+1}^\mu &= (q_n^\mu + V_n \Delta t H'(p_n^\mu))_+ \end{cases} \quad p_0^\mu = \mu \quad q_0^\mu = q_0$$

and then to find numerically a minimum to the function

$$\mu \leq 0 \mapsto \mathcal{I}_N(q^\mu) = \sum_{n=0}^{N-1} V_n L\left(\frac{q_n^\mu - q_{n+1}^\mu}{V_n \Delta t}\right) + \frac{1}{2} \gamma \sigma^2 q_n^{\mu 2},$$

\mathcal{I}_N being a discrete counterpart of \mathcal{I} .

6.3 The discrete problem

An approach that is similar to the above one consists in considering the function

$$(q_n)_{0 \leq n \leq N} \mapsto \mathcal{I}_N(q) = \sum_{n=0}^{N-1} V_n L \left(\frac{q_n - q_{n+1}}{V_n \Delta t} \right) + \frac{1}{2} \gamma \sigma^2 q_n^2,$$

and to find a minimum to this function while imposing $q_0 = q_0$ and $q_N = 0$.

This leads to the following equation (that is obviously a discretization of the Euler-Lagrange equation):

$$\forall 1 \leq n \leq N-1, \quad L' \left(\frac{q_n - q_{n+1}}{V_n \Delta t} \right) - L' \left(\frac{q_{n-1} - q_n}{V_{n-1} \Delta t} \right) + \gamma \sigma^2 \Delta t q_n = 0, \quad q_0 = q_0, q_N = 0.$$

Then, the backward shooting approach consists in introducing a sequence $(q_n^\nu)_{0 \leq n \leq N}$ with $q_N^\nu = 0$, $q_{N-1}^\nu = \nu > 0$ and:

$$q_{n-1}^\nu = q_n^\nu + V_{n-1} \Delta t H' \left(L' \left(\frac{q_n^\nu - q_{n+1}^\nu}{V_n \Delta t} \right) + \gamma \sigma^2 \Delta t q_n^\nu \right).$$

q_0^ν is a continuous and strictly increasing function of ν and there exists a unique $\nu > 0$ such that $q_0^\nu = q_0$. However, for the same reason as above, the appropriate ν can be really small and unreachable numerically. In such a case, we recommend to use the method presented above with the Hamiltonian system.

6.4 Hamilton-Jacobi equation

The preceding methods can be used to approximate the optimal liquidation strategy numerically. Then, if we are interested in the price of a block trade, we can approximate it using the function \mathcal{I}_N . It is the discrete counterpart of the two-step approach of Section 4. Instead, the direct approach can be used by computing numerically $\theta_{K,T}$ for K large.

To that purpose, we refer to the classical literature on monotone finite difference schemes for first order Hamilton-Jacobi equations. In our case, a scheme to approximate $\theta_{K,T}$ on the grid $\{(t_n, x_m), 0 \leq n \leq N, 0 \leq m \leq M\}$ where $t_0 = 0 < t_1 < \dots < t_n = n\Delta t < \dots < t_N = T$ and $x_0 = 0 < x_1 < \dots < x_m = m\Delta x < \dots < x_M = M\Delta x = q_0$ can be:

$$\theta_{K,T}^{n-1,m} = \theta_{K,T}^{n,m} + \frac{1}{2} \gamma \sigma^2 \Delta t x_m^2 - V_n \Delta t H \left(\left(\frac{\theta_{K,T}^{n,m} - \theta_{K,T}^{n,m-1}}{\Delta x} \right)_+ \right), \quad 1 \leq n \leq N, 1 \leq m \leq M$$

$$\theta_{K,T}^{N,m} = Kx_m^2, \quad 0 \leq m \leq M \quad \theta_{K,T}^{n,0} = 0, \quad 0 \leq n \leq N,$$

where Δt and Δx are so that $V_n \frac{\Delta t}{\Delta x} H' \left(\left(\frac{\theta_{K,T}^{n,m} - \theta_{K,T}^{n,m-1}}{\Delta x} \right)_+ \right) \leq 1$, $1 \leq n \leq N, 1 \leq m \leq M$.

Conclusion

This article was motivated by the lack of a general framework for liquidation problems and by the need to give a price to block trades. We developed such a general framework and proved existence and regularity results for optimal liquidation strategies. We provided two different proofs that considering stochastic liquidation strategies does not improve the result obtained with the best deterministic liquidation strategy. Regarding block trade pricing, we provide completely new results that now allow to quantify liquidity premia. Specifically, we provide a closed-form formula for the price a block trade when there is no time constraint to liquidate, and a differential characterization in the time-constrained case.

We also complemented our results with important remarks concerning the numerical approximations of optimal liquidation strategies and block trade prices.

The results we obtained can be generalized straightforwardly when one introduces a constant drift in the price process. Generalizing the results to the case of a multi-asset portfolio is currently a work in progress.

References

- [1] R. Almgren. Optimal trading with stochastic liquidity and volatility. *SIAM Journal of Financial Mathematics*, to appear, 2011.
- [2] R. Almgren and N. Chriss. Value under liquidation. *Risk*, 12(12):61–63, 1999.
- [3] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.
- [4] R. Almgren and J. Lorenz. Adaptive arrival price. *Journal of Trading*, 2007(1):59–66, 2007.
- [5] R.F. Almgren. Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied Mathematical Finance*, 10(1):1–18, 2003.
- [6] E. Bayraktar and M. Ludkovski. Liquidation in limit order books with controlled intensity. *Mathematical Finance*, 2012.

- [7] D. Bertsimas and A. Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1(1):1–50, 1998.
- [8] B. Bouchard, N.M. Dang, and C.-A. Lehalle. Optimal control of trading algorithms: a general impulse control approach. 2009.
- [9] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58. Birkhäuser Boston, 2004.
- [10] L.C. Evans. Partial Differential Equations (Graduate Studies in Mathematics, Vol. 19). 2009.
- [11] P. A. Forsyth, J. S. Kennedy, S. T. Tse, and H. Windcliff. Optimal trade execution: a mean quadratic variation approach. *Quantitative Finance*, 2009.
- [12] P.A. Forsyth. A Hamilton Jacobi Bellman approach to optimal trade execution. *Applied Numerical Mathematics*, 2010.
- [13] J. Gatheral. No-dynamic-arbitrage and market impact. *Quantitative Finance*, 10(7):749–759, 2010.
- [14] J. Gatheral and A. Schied. Optimal trade execution under geometric brownian motion in the almgren and chris framework. *International Journal of Theoretical and Applied Finance*, 14(03):353–368, 2011.
- [15] R.C. Grinold and R.N. Kahn. Active portfolio management. 1999.
- [16] O. Guéant and C.-A. Lehalle. General intensity shapes in optimal liquidation. *Working Paper*, 2012.
- [17] O. Guéant, C.-A. Lehalle, and J. Fernandez Tapia. Optimal portfolio liquidation with limit orders. *SIAM Journal of Financial Mathematics*, to appear, 2012.
- [18] P. Kratz and T. Schöneborn. Optimal liquidation in dark pools. 2009.
- [19] P. Kratz and T. Schöneborn. Portfolio liquidation in dark pools in continuous time. 2012.
- [20] J. Lorenz and R. Almgren. Mean-Variance Optimal Adaptive Execution. *Applied Mathematical Finance*, To appear in 2011.
- [21] A. Obizhaeva and J. Wang. Optimal trading strategy and supply/demand dynamics. Technical report, National Bureau of Economic Research, 2005.
- [22] R.T. Rockafellar. *Convex analysis*, volume 28. Princeton university press, 1996.
- [23] A. Schied and T. Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13(2):181–204, 2009.

- [24] A. Schied, T. Schöneborn, and M. Tehranchi. Optimal basket liquidation for cara investors is deterministic. *Applied Mathematical Finance*, 17(6):471–489, 2010.
- [25] T. Schöneborn. Adaptive basket liquidation. 2009.
- [26] S. T. Tse, P. A. Forsyth, J. S. Kennedy, and H. Windcliff. Comparison between the mean variance optimal and the mean quadratic variation optimal trading strategies. 2011.